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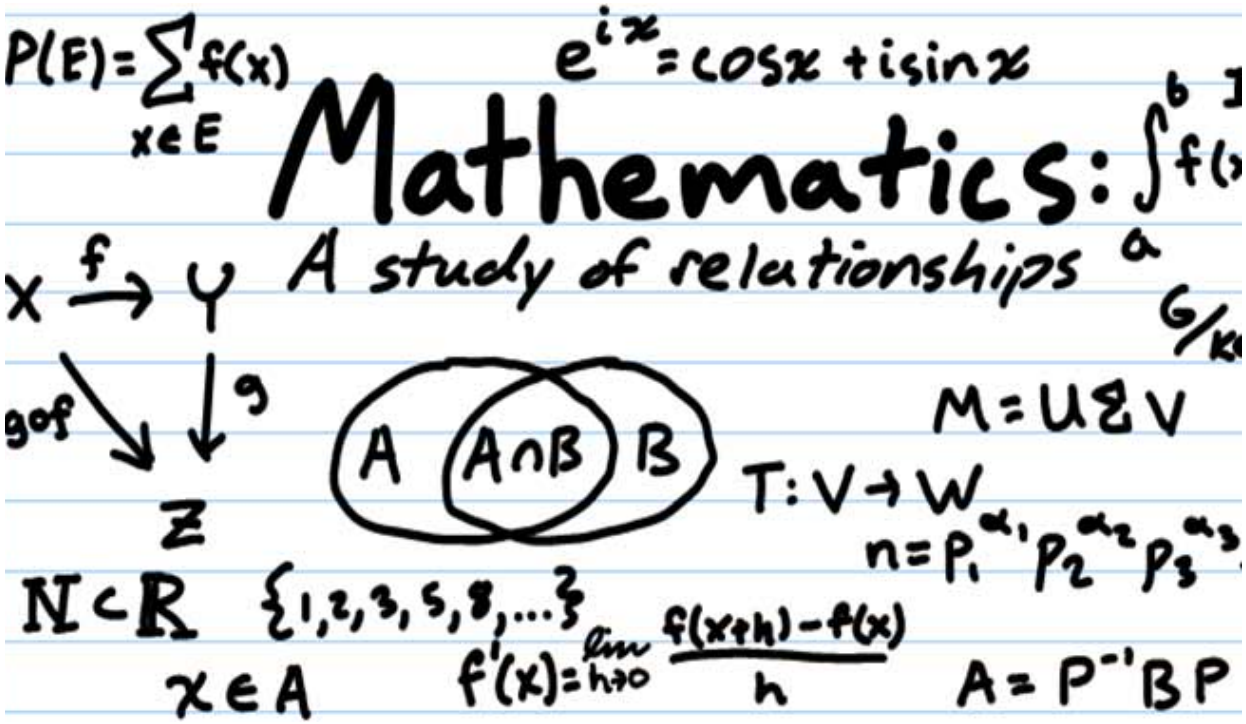
MATERIA: Matematica

# Esercizi di ripasso



# Funzioni: definizione, proprietà, caratteristiche

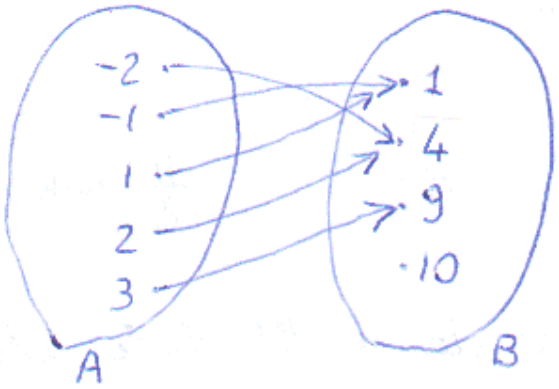
$A = \{-2; -1; 1; 2; 3\}$

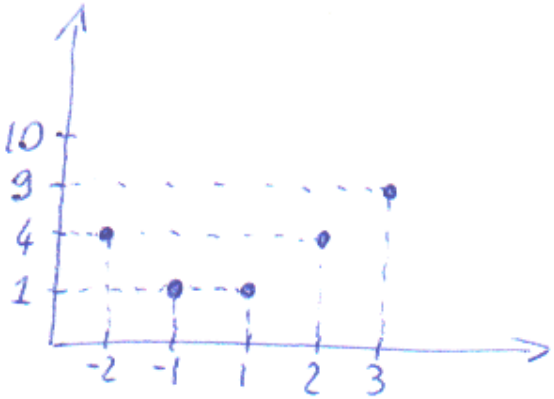
$B = \{1; 4; 9; 10\}$

$p(x;y) =$ "$y = x^2$" $\quad x \in A,\ y \in B$

$-2\,R\,4$
$-1\,R\,1$
$1\,R\,1$
$2\,R\,4$
$3\,R\,9$

La relazione è una funzione perché ad ogni elemento di A corrisponde uno ed un solo elemento di B.



$A = \{0; 1; 2; 3\}$

$f : A \to \mathbb{Z}$

$f(x) = 2x - 3$

$C = ?$ codominio

Il codominio è l'insieme di tutte le immagini.

$f(0) = 2 \cdot 0 - 3 = -3$

$f(1) = 2 \cdot 1 - 3 = -1$

$f(2) = 2 \cdot 2 - 3 = 1$

$f(3) = 3 \cdot 2 - 3 = 3$

$C = \{-3; -1; 1; 3\}$

$p(x;y) =$ " $y = \frac{|x|}{x}$ " $\quad x \in \mathbb{Q}, \quad y \in \mathbb{Q}$

La relazione non è una funzione perché ad $x=0$ non corrisponde nessun $y \in \mathbb{Q}$.

$$f(x) = \frac{|x|}{x} = \begin{cases} \frac{x}{x} = 1 & \text{se } x > 0 \\ -\frac{x}{x} = -1 & \text{se } x < 0 \end{cases}$$

$x \in \mathbb{R}$

Il denominatore di $\frac{|x|}{x}$ deve essere diverso da 0.

Quindi il dominio è

$$D = \mathbb{R} - \{0\}$$

Il codominio è l'insieme di tutte le immagini:

$$C = \{-1; 1\}$$

$f : \mathbb{N} \to \mathbb{Q}$

$f(x) = \frac{2x-1}{2}$

immagine di 5 ?

$f(5) = \frac{2 \cdot 5 - 1}{2} = \frac{9}{2}$

verificare che 5 non ha controimmagine

$y = 5 \qquad x = ?$

$y = \frac{2x-1}{2}$

$5 = \frac{2x-1}{2}$

$10 = 2x - 1$

$11 = 2x \qquad x = \frac{11}{2}$ $\qquad \frac{11}{2}$ non è un numero naturale

Non esiste $x \in \mathbb{N}$ tale che $f(x) = 5$.

$f(x) = \frac{1}{2}x - 4$

$f : \mathbb{R} \to \mathbb{R}$

immagine di $\frac{3}{2}$ ?

$x = \frac{3}{2}$ $\quad y = f\left(\frac{3}{2}\right) = \frac{1}{2} \cdot \frac{3}{2} - 4 = \frac{3}{4} - 4 = \frac{3-16}{4} = -\frac{13}{4}$

controimmagine di $-\frac{1}{6}$ ?

$y = -\frac{1}{6}$ $\quad x = ?$

$y = \frac{1}{2}x - 4$

$-\frac{1}{6} = +\frac{1}{2}x - 4 \longrightarrow -1 = 3x - 24 \longrightarrow 23 = 3x$

$x = \frac{23}{3}$

$f(x) = 2x^2 + 3x - 4$

$x \in \mathbb{R}$

immagine di 0?

$x = 0$

$y = f(0) = 2 \cdot 0^2 + 3 \cdot 0 - 4 = -4$

controimmagini di 10?

$y = 10 \qquad x = ?$

$y = 2x^2 + 3x - 4$

$10 = 2x^2 + 3x - 4$

$2x^2 + 3x - 14 = 0$

$\Delta = 9 + 112 = 121 = 11^2$

$$x_{1,2} = \frac{-3 \pm 11}{4} = \begin{cases} - & -\frac{14}{4} = -\frac{7}{2} \\ + & \frac{8}{4} = 2 \end{cases}$$

$$f(x) = \begin{cases} x+3 & \text{per } x \geq 1 \\ -x^2+x & \text{per } x \leq 1 \end{cases}$$

$f_1(x) = x+3$

$f_2(x) = -x^2+x$

per $x=1$

$f_1(1) = 1+3 = 4$

$f_2(1) = -(1)^2+1 = 0$

la legge non definisce una funzione perché ad $x=1$ sono associati due elementi di B ( $y=4$ e $y=0$).

$f : \mathbb{Z} \to \mathbb{Z}$

$f(x) = ax + b$

$a = ?$

$b = ?$

$f(1) = 7$

$f(-2) = 1$

$f(1) = a + b$

$f(-2) = -2a + b$

$$\begin{cases} a + b = 7 \\ -2a + b = 1 \end{cases} \to \begin{cases} a = 7 - b \\ -14 + 2b + b = 1 \end{cases} \to \begin{cases} a = 7 - b \\ 3b = 15 \end{cases}$$

$$\to \begin{cases} a = 2 \\ b = 5 \end{cases}$$

$f(x) = 2x + 5$

$f : \mathbb{N} \to \mathbb{N}$

$f(x) = 2(x+1)+3$

immagine di 2 ?

$x = 2$

$f(2) = 2(2+1)+3 = 2 \cdot 3 + 3 = 6+3 = 9$

controimmagine di 13 ?

$y = 13 \qquad x = ?$

$y = 2(x+1)+3$

$13 = 2x+2+3$

$13 = 2x+5$

$8 = 2x$

$x = 4$

$f: \mathbb{Q} \to \mathbb{Q}$

$f(x) = x^2 - 4x + 6$

immagine di 0 ?

$x = 0 \qquad y = ?$

$y = f(0) = 0^2 - 4 \cdot 0 + 6 = 6$

immagine di 1 ?

$x = 1 \qquad y = ?$

$y = f(1) = 1^2 - 4 \cdot 1 + 6 = 3$

controimmagini di 4 ?

$y = 4 \qquad x = ?$

$y = x^2 - 4x + 6$

$4 = x^2 - 4x + 6$

$x^2 - 4x + 2 = 0$

$\Delta = 16 - 8 = 8$

$$x_{1,2} = \frac{4 \pm \sqrt{8}}{2} = \frac{4 \pm 2\sqrt{2}}{2} = \begin{cases} 2 - \sqrt{2} \\ 2 + \sqrt{2} \end{cases}$$

$2 - \sqrt{2}$ è un numero irrazionale

$2 + \sqrt{2}$ " " " "

4 non ha controimmagini in $\mathbb{Q}$

$f : \mathbb{Z} \to \mathbb{Z}$ $\quad f(x) = x^8$

$g : \mathbb{Z} \to \mathbb{Z}$ $\quad g(x) = 5x^4 - 4$

$x = ?$ $\quad f(x) = g(x)$

$$f(x) = g(x)$$

$$x^8 = 5x^4 - 4$$

$$x^8 - 5x^4 + 4 = 0$$

$$(x^4 - 4)(x^4 - 1) = 0$$

$$(x^2+2)(x^2-2)(x^2+1)(x^2-1) = 0$$

$x = \pm\sqrt{2}$ non appartengono a $\mathbb{Z}$

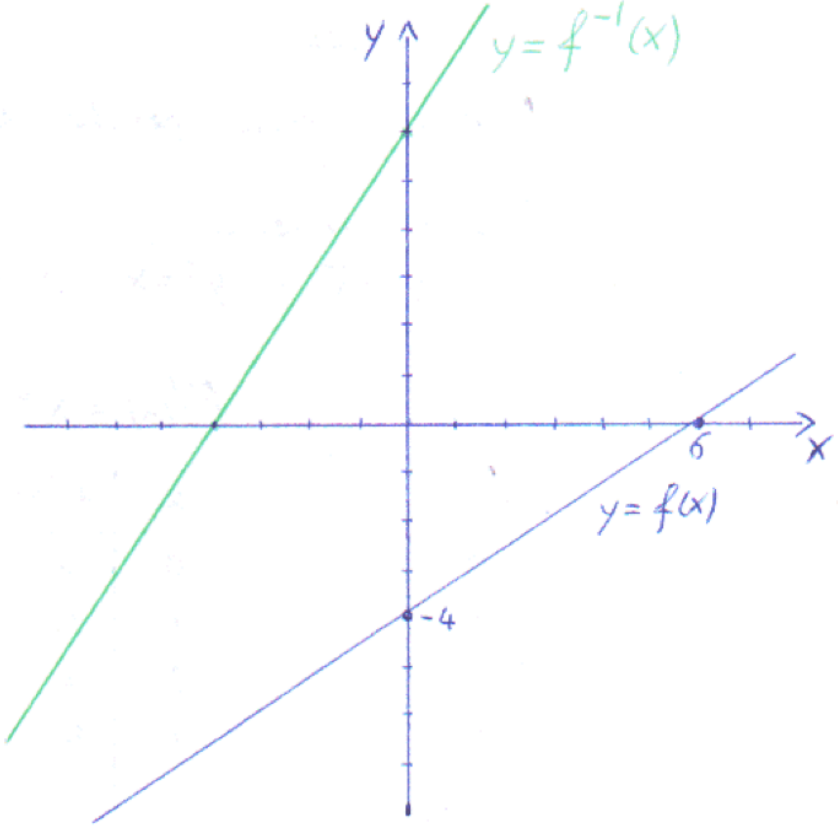
$x = \pm 1$

$f(x) = \frac{2}{3}x - 4$

| $x$ | $f(x)$ |
|---|---|
| -6 | -8 |
| -3 | -6 |
| 0 | -4 |
| 3 | -2 |
| 6 | 0 |
| 9 | 2 |
| 12 | 4 |

$f^{-1}(x) = \frac{3}{2}x + 6$

| $x$ | $f^{-1}(x)$ |
|---|---|
| -6 | -3 |
| -4 | 0 |
| -2 | 3 |
| 0 | 6 |
| 2 | 9 |
| 4 | 12 |



$f: \mathbb{R} \to \mathbb{R}$

$f(x) = \frac{2}{3}x - 4$

$f$ è iniettiva se $f(x_1) = f(x_2) \to x_1 = x_2$

$$f(x_1) = f(x_2) \to \frac{2}{3}x_1 - 4 = \frac{2}{3}x_2 - 4 \to \frac{2}{3}x_1 = \frac{2}{3}x_2 \to x_1 = x_2$$

$f$ è suriettiva se $\forall y \in \mathbb{R}, \exists x \in \mathbb{R} \mid f(x) = y$

$$y = \frac{2}{3}x - 4 \to y + 4 = \frac{2}{3}x \to \frac{3y+12}{\cancel{3}} = \frac{2x}{\cancel{3}} \to x = \frac{3}{2}y + 6$$

$f$ è biunivoca perché è iniettiva e suriettiva

$$y = \frac{2}{3}x - 4 \to y + 4 = \frac{2}{3}x \to x = \frac{3}{2}y + 6$$

$x \to y$
$y \to x$

$$y = \frac{3}{2}x + 6 \to f^{-1}(x) = \frac{3}{2}x + 6$$

$f(x) = \sqrt{x} \qquad x \in \mathbb{R}_0^+$

$f$ è iniettiva:

$$f(x_1) = f(x_2) \rightarrow \sqrt{x_1} = \sqrt{x_2} \rightarrow x_1 = x_2$$

$f$ è suriettiva:

$$y = \sqrt{x} \rightarrow x = y^2 \qquad \text{codominio} \quad C = \mathbb{R}_0^+$$

$f$ è biunivoca perché è iniettiva e suriettiva

funzione inversa:

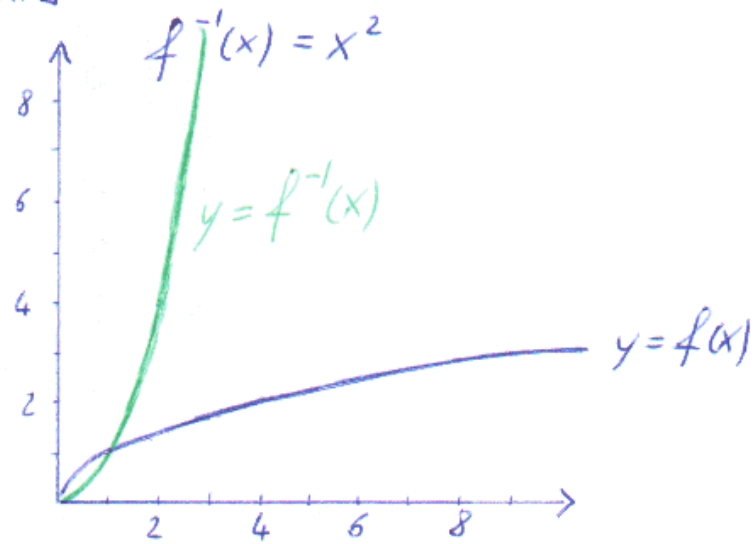
$$y = \sqrt{x} \rightarrow y^2 = x \rightarrow x = y^2 \qquad \begin{bmatrix} x \rightarrow y \\ y \rightarrow x \end{bmatrix} \qquad y = x^2$$

$f(x) = \sqrt{x}$

| $x$ | $f(x)$ |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 4 | 2 |
| 9 | 3 |

$f^{-1}(x) = x^2$

| $x$ | $f^{-1}(x)$ |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 2 | 4 |
| 3 | 9 |



$f(x) = -\sqrt{x} \qquad x \in \mathbb{R}_0^+$

$f$ è iniettiva:

$f(x_1) = f(x_2) \rightarrow -\sqrt{x_1} = -\sqrt{x_2} \rightarrow \sqrt{x_1} = \sqrt{x_2} \rightarrow x_1 = x_2$

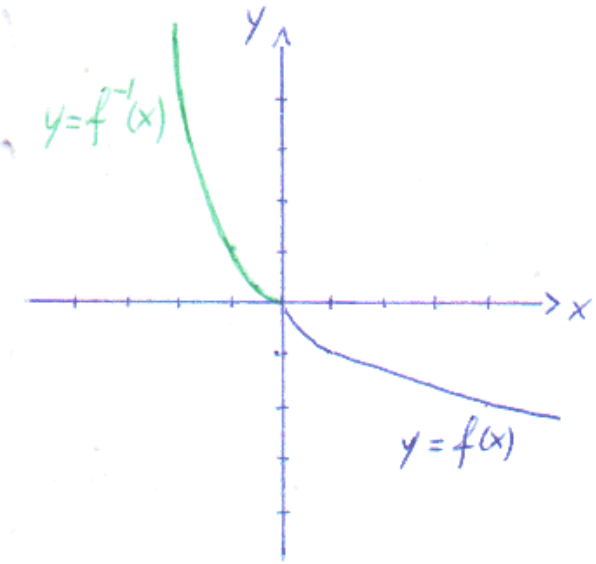
$f$ è suriettiva:

$y = -\sqrt{x} \rightarrow -y = \sqrt{x} \rightarrow x = (-y)^2 \qquad y \leq 0$

$f$ è biunivoca e invertibile:

$y = -\sqrt{x} \rightarrow x = (-y)^2 \rightarrow \begin{bmatrix} x \rightarrow y \\ y \rightarrow x \end{bmatrix} \quad y = (-x)^2 \rightarrow$

$\rightarrow y = x^2. \qquad f^{-1}(x) = x^2 \qquad x \in \mathbb{R}_0^-$



$f : \mathbb{Q} \to \mathbb{Q}$

$f(x) = \frac{3x-1}{x}$ $\quad x \neq 0 \quad$ dominio $\quad D = \mathbb{Q} - \{0\}$

$y = \frac{3x-1}{x} \to x \neq 0 \quad xy = 3x-1 \to \quad xy - 3x = -1 \to \quad 3x - xy = 1 \to$

$\to x(3-y) = 1 \to \quad y \neq 3 \quad x = \frac{1}{3-y}$

codominio $\quad C = \mathbb{Q} - \{3\}$

$f(x_1) = f(x_2) \to \frac{3x_1 - 1}{x_1} = \frac{3x_2 - 1}{x_2} \to \cancel{3x_1x_2} - x_2 = \cancel{3x_1x_2} - x_1 \to x_1 = x_2$

$f$ è iniettiva

funzione inversa:

$y = \frac{3x-1}{x} \to xy = 3x-1 \to \quad 1 = 3x - xy \to \quad 1 = x(3-y) \to$

$\to x = \frac{1}{3-y} \qquad \begin{bmatrix} x \to y \\ y \to x \end{bmatrix} \qquad y = \frac{1}{3-x} \qquad f^{-1}(x) = \frac{1}{3-x}$

$A = (0;1] = \{x \in \mathbb{R} \mid 0 < x \leq 1\}$

$f : \mathbb{R}_0^+ \to A$

$f(x) = \frac{1}{1+x^2}$

$f(x_1) = f(x_2) \to \frac{1}{1+x_1^2} = \frac{1}{1+x_2^2} \to 1+x_2^2 = 1+x_1^2 \to$

$\to x_2^2 = x_1^2 \to x_1 = x_2$ $\quad$ $f$ è iniettiva

$y = \frac{1}{1+x^2} \to y + x^2 y = 1 \to x^2 y = 1-y \to$

$y \neq 0 \quad x^2 = \frac{1-y}{y} \to$ se $y \in A \quad x = \sqrt{\frac{1-y}{y}}$

$f$ è suriettiva

$f$ è biunivoca e invertibile:

$x = \sqrt{\frac{1-y}{y}} \qquad \begin{bmatrix} x \to y \\ y \to x \end{bmatrix} \qquad y = \sqrt{\frac{1-x}{x}}$

$f^{-1}(x) = \sqrt{\frac{1-x}{x}}$

$x \in \mathbb{Z}$

$f(x) = 2x+3$

$g(x) = 3x-1$

$(f \circ g)(x) = f(g(x)) = f(3x-1) = 2(3x-1)+3 = 6x-2+3 = 6x+1$

$(g \circ f)(x) = g(f(x)) = g(2x+3) = 3(2x+3)-1 = 6x+9-1 = 6x+8$

$f \circ g \neq g \circ f$ perché $6x+1 \neq 6x+8$

$y = 26$ $\quad 6x+1 = 26 \rightarrow 6x = 25 \rightarrow x = \frac{25}{6} \notin \mathbb{Z}$

26 non ha controimmagine in $f \circ g$

$y = 26$ $\quad 6x+8 = 26 \rightarrow 6x = 18 \rightarrow x = 3$

26 ha controimmagine $x = 3$ in $g \circ f$

$f(x) = x+1$ $\quad x \in \mathbb{R}$

$g(x) = 2x-1$

$(f \circ g)(x) = f(2x-1) = 2x-1+1 = 2x$

$(g \circ f)(x) = g(x+1) = 2(x+1)-1 = 2x+2-1 = 2x+1$

$(f \circ g)(x) = (g \circ f)(x)$

$2x = 2x+1 \rightarrow 0x = 1$ impossibile

$f: \mathbb{Z} \to \mathbb{Z}$ $\quad f(x) = x^2 - x$

$g: \mathbb{Z} \to \mathbb{Z}$ $\quad g(x) = 2x$

$a(x) = (f \circ g)(x) = f(2x) = 4x^2 - 2x$

$b(x) = (g \circ f)(x) = g(x^2 - x) = 2x^2 - 2x$

$f \circ g \neq g \circ f$

$f(x) = 2x + 5$

$f: \mathbb{R} \to \mathbb{R}$

$f$ è iniettiva:

$f(x_1) = f(x_2) \to 2x_1 + 5 = 2x_2 + 5 \to 2x_1 = 2x_2 \to x_1 = x_2$

$f$ è suriettiva:

$y = 2x + 5 \to y - 5 = 2x \to x = \frac{y-5}{2}$

$f$ è biunivoca perché è iniettiva e suriettiva

$f$ è una funzione crescente:

$\forall x_1, x_2 \in \mathbb{R}, \quad x_1 < x_2 \to f(x_1) < f(x_2)$

$x_1 < x_2 \to 2x_1 < 2x_2 \to 2x_1 + 5 < 2x_2 + 5 \to f(x_1) < f(x_2)$

$f : \mathbb{R} \to \mathbb{R}$

$f(x) = 2x + \frac{1}{2}$

$$f(x_1) = f(x_2) \to 2x_1 + \cancel{\tfrac{1}{2}} = 2x_2 + \cancel{\tfrac{1}{2}} \to 2x_1 = 2x_2 \to x_1 = x_2$$

$f$ è iniettiva

$$y = 2x + \frac{1}{2} \to y - \frac{1}{2} = 2x \to x = \frac{1}{2}y - \frac{1}{4}$$

$f$ è suriettiva

$f$ è biunivoca perché è iniettiva e suriettiva

funzione inversa:

$$y = 2x + \frac{1}{2} \to y - \frac{1}{2} = 2x \to x = \frac{1}{2}y - \frac{1}{4}$$

$$\to \begin{bmatrix} x \to y \\ y \to x \end{bmatrix} \qquad y = \frac{1}{2}x - \frac{1}{4} \qquad f^{-1}(x) = \frac{1}{2}x - \frac{1}{4}$$

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f\left(\frac{1}{2}x - \frac{1}{4}\right) = 2\left(\frac{1}{2}x - \frac{1}{4}\right) + \frac{1}{2} =$$

$$= x - \frac{1}{2} + \frac{1}{2} = x$$

$f(x) = 3-2x$

$f$ è iniettiva:

$f(x_1) = f(x_2) \rightarrow 3-2x_1 = 3-2x_2 \rightarrow -2x_1 = -2x_2 \rightarrow x_1 = x_2$

$f$ è suriettiva:

$y = 3-2x \rightarrow 2x = 3-y \rightarrow x = \frac{3-y}{2}$

$f$ è biunivoca perché è iniettiva e suriettiva

$f$ è una funzione decrescente:

$\forall x_1, x_2 \in \mathbb{R}, \quad x_1 < x_2 \rightarrow f(x_1) > f(x_2)$

$x_1 < x_2 \rightarrow -2x_1 > -2x_2 \rightarrow 3-2x_1 > 3-2x_2 \rightarrow f(x_1) > f(x_2)$

$f(x) = x^3 + 1$

$f$ è crescente:

$\forall x_1, x_2 \in \mathbb{R}, \quad x_1 < x_2 \rightarrow f(x_1) < f(x_2)$

$$x_1 < x_2 \rightarrow x_1^3 < x_2^3 \rightarrow x_1^3 + 1 < x_2^3 + 1 \rightarrow f(x_1) < f(x_2)$$

$f$ è iniettiva:

$$f(x_1) = f(x_2) \rightarrow x_1^3 + 1 = x_2^3 + 1 \rightarrow x_1^3 = x_2^3 \rightarrow x_1 = x_2$$

$f$ è suriettiva:

$$y = x^3 + 1 \rightarrow x^3 = y - 1 \rightarrow x = \sqrt[3]{y-1}$$

$f$ è biunivoca e invertibile:

$$y = x^3 + 1 \rightarrow x^3 = y - 1 \rightarrow x = \sqrt[3]{y-1}$$

$$\begin{bmatrix} x \rightarrow y \\ y \rightarrow x \end{bmatrix} \qquad y = \sqrt[3]{x-1}$$

$$h(x) = \sqrt[3]{x-1}$$

$f(x) = \frac{x}{x-1}$

$f$ è decrescente in $(-\infty; 1)$

$x_1 < 1 \quad , \quad x_2 < 1$

$x_1 < x_2 \rightarrow \quad x_1 - x_1 x_2 < x_2 - x_1 x_2 \rightarrow x_1 (1 - x_2) < x_2 (1 - x_1) \rightarrow$

$\rightarrow \frac{x_1}{1-x_1} < \frac{x_2}{1-x_2} \rightarrow -\frac{x_1}{x_1-1} < -\frac{x_2}{x_2-1} \rightarrow \frac{x_1}{x_1-1} > \frac{x_2}{x_2-1} \rightarrow$

$\rightarrow f(x_1) > f(x_2)$

$f$ è decrescente in $(1; +\infty)$

$x_1 > 1 \quad , \quad x_2 > 1$

$x_1 < x_2 \rightarrow x_1 - x_1 x_2 < x_2 - x_1 x_2 \rightarrow -x_1 + x_1 x_2 > -x_2 + x_1 x_2 \rightarrow$

$\rightarrow x_1 (x_2 - 1) > x_2 (x_1 - 1) \rightarrow \frac{x_1}{x_1-1} > \frac{x_2}{x_2-1} \rightarrow f(x_1) > f(x_2)$

funzione inversa:

$y = \frac{x}{x-1} \rightarrow x \neq 1 \qquad yx - y = x \rightarrow yx - x = y \rightarrow x(y-1) = y \rightarrow$

$\rightarrow y \neq 1 \quad x = \frac{y}{y-1} \rightarrow \begin{bmatrix} x \rightarrow y \\ y \rightarrow x \end{bmatrix} \quad y = \frac{x}{x-1}$

$g(x) = \frac{x}{x-1}$

# Dominio
# di una funzione

*Determinare il dominio della funzione*

$$y = \frac{x^4 - 1}{x^3 + x^2 - 10x + 8}$$

Anche in questo caso bisogna porre il denominatore diverso da zero

$$x^3 + x^2 - 10x + 8 \neq 0$$

Utlizzando la *regola di Ruffini* è possibile scomporre il polinomio di terzo grado nel prodotto dei due fattori

$$x^3 + x^2 - 10x + 8 = (x - 1)\left(x^2 + 2x - 8\right)$$

*(Il secondo fattore potrebbe essere scomposto ancora con il metodo di* Ruffini*)*
Il dominio $D$ della funzione si determina ponendo

$$\begin{cases} x - 1 \neq 0 \\ x^2 + 2x - 8 \neq 0 \end{cases}$$

La prima inequazione dà come risultato

$$x \neq 1 \tag{4}$$

Risolviamo l'inequazione $x^2 + 2x - 8 \neq 0$ utilizzando la formula ridotta

$$\frac{\Delta}{4} = 1 + 8 = 9$$

$$x_{1,2} \neq -1 \pm 3$$

da cui

$$x_1 \neq -4 \tag{5}$$

$$x_1 \neq 2 \tag{6}$$

La soluzione formata dalle (4), (5) e (6) fornisce anche il dominio della funzione:

$$D = \{x \in \mathbb{R} | x \neq -4 \vee x \neq 1 \vee x \neq 2\}$$

$$D = \mathbb{R} \setminus \{-4, 1, 2\}$$

*Determinare il dominio della funzione*

$$y = \sqrt{\frac{x^2 - 4x}{1 - x^2}}$$

Anche in questo caso, trattandosi di una funzione irrazionale, dovremo porre il radicando maggiore o uguale a zero e studiare la disequazione:

$$\frac{x^2 - 4x}{1 - x^2} \geq 0 \tag{8}$$

si tratta di una *disequazione fratta* da studiare ponendo maggiore o uguale a zero il numeratore $(N)$ e maggiore a zero il denominatore $(D)$ per poi studiare il prodotto dei segni:

$$N \geq 0; \qquad x^2 - 4x \geq 0; \qquad x\,(x - 4) \geq 0$$

da cui

$$N_1 \geq 0; \qquad x \geq 0$$

$$N_2 \geq 0; \qquad x - 4 \geq 0; \qquad x \geq 4$$

Studiamo il segno del denominatore:

$$D > 0; \qquad 1 - x^2 > 0; \qquad x^2 - 1 < 0$$

Dall'equazione associata ricaviamo le due soluzione (vedi esercizio precedente) $x_1 = -1$ e $x_2 = 1$, ma essendo questa volta il verso della disequazione discorde rispetto al coefficiente di $x^2$ si avrà:

$$-1 < x < 1$$

Ricapitolando

$$N_1 \geq 0; \qquad x \geq 0$$

$$N_2 \geq 0; \qquad x - 4 \geq 0; \qquad x \geq 4$$

$$D > 0; \qquad -1 < x < 1$$

Non essendoci soluzioni di molteplicità *pari* [2] si avrà alternanza di segni negli intervalli individuati dalle soluzioni stesse, ed assumendo la frazione valore *negativo* per $x = -2$, valore minore della soluzione più piccola, si avrà la seguente alternanza dei segni negli intervalli individuati dalle soluzioni:

$$\text{la frazione} \quad \frac{x^2 - 4x}{1 - x^2} \quad \text{assume segno} \quad \begin{cases} - & \text{se } x \in ]-\infty; -1[ \\ + & \text{se } x \in ]-1; 0[ \\ - & \text{se } x \in ]0; 1[ \\ + & \text{se } x \in ]1; 4[ \\ - & \text{se } x \in ]4; +\infty[ \end{cases}$$

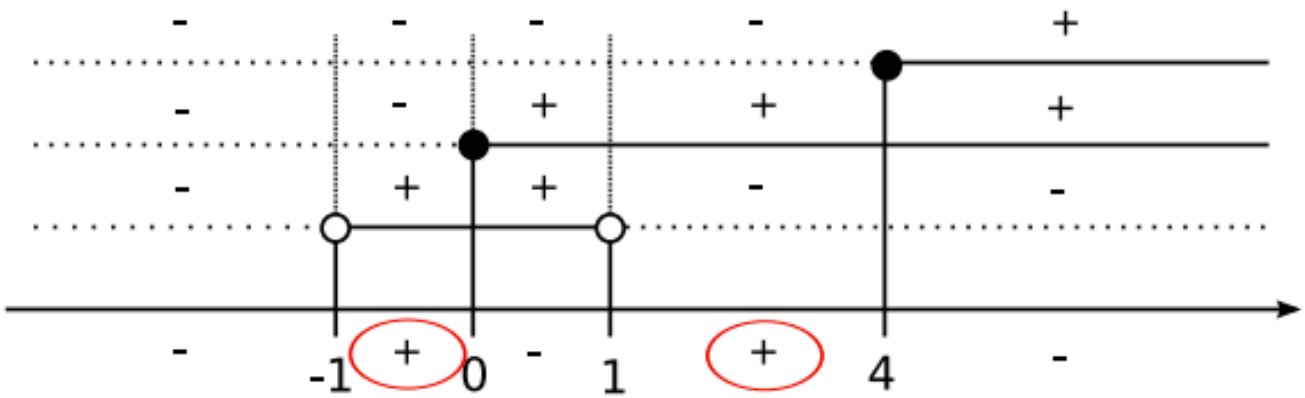
Confrontando i segni degli intervalli con il verso della disequazione (8) si può concludere che il dominio della funzione è:

$$D = \{x \in \mathbb{R} | -1 < x \leq 0 \vee 1 < x \leq 4\}$$

$$D = ]-1; 0] \cup ]1 : 4]$$

essendo $x = 0$ e $x = 4$ inclusi in quanto soluzioni del numeratore.

Verifichiamolo graficamente



*Determinare il dominio della funzione*

$$y = \frac{\sqrt{x^2 - 3}}{x - 5} + \sqrt{-x^2 + x + 30}$$

In questo caso abbiamo due radici e un termine frazionario. Le tre condizioni vanno messe a sistema:

$$\begin{cases} x^2 - 3 \geq 0 \\ x - 5 \neq 0 \\ x^2 - x - 30 \leq 0 \end{cases} \tag{10}$$

Per la seconda radice abbiamo cambiato il segno di tutti i termini e il verso della disequazione. Risolviamo la prima disequazione della (10). Consideriamo l'equazione associata:

$$x^2 - 3 = 0$$

$$x^2 = 3$$

da cui si ottengono le **due soluzioni opposte**

$$x = -\sqrt{3}$$

$$D = \{x \in \mathbb{R} | -2 \leq x \leq 1\}$$

$$D = [-2; 1]$$

$$x = \sqrt{3}$$

Avendo trovato **due** soluzioni **distinte** il $\Delta$ associato è positivo per cui la soluzione della disequazione (10) è data dall'insieme degli *intervalli esterni* rispetto alle due soluzioni ottenute

$$x \leq -\sqrt{3} \vee x \geq \sqrt{3} \tag{11}$$

L'inequazione $x - 5 \neq 0$ ha soluzione

$$x \neq 5 \tag{12}$$

L'ultima disequazione è di secondo grado *completa* :

$$\Delta = 1 + 120 = 121$$

$$x_{1,2} = \frac{1 \pm \sqrt{121}}{2} = \frac{1 \pm 11}{2}$$

Le soluzioni dell'equazione associata sono:

$$x_1 = \frac{1 - 11}{2} = \frac{-10}{2} = -5$$

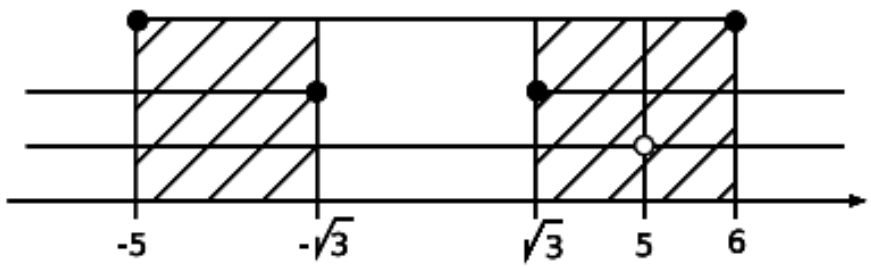
e

$$x_2 = \frac{1 + 11}{2} = \frac{12}{2} = 6$$

La soluzione della disequazione $x^2 - x - 30 \leq 0$ sarà dunque:

$$-5 \lneq x \leq 6 \tag{13}$$

Riportiamo la (10), la 12 e la (13) sul grafico del sistema:



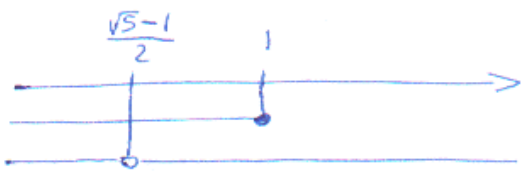

da cui ricaviamo il dominio:

$$D = \left\{ x \in \mathbb{R} | -5 \leq x \leq -\sqrt{3} \vee \sqrt{3} \leq x \leq 6 \vee x \neq 5 \right\}$$

$$D = \left[ -5; -\sqrt{3} \right] \cup \left[ \sqrt{3}; 5 \left[ \cup \right] 5; 6 \right]$$

$$\boxed{y = \frac{2x^2 + \sqrt{1-x^3}}{x - \sqrt{1-x}}}$$

$$\begin{cases} 1-x^3 \geq 0 \\ 1-x \geq 0 \\ x-\sqrt{1-x} \neq 0 \end{cases} \qquad \begin{cases} (1-x)\cancel{(1+x+x^2)} \geq 0 \\ x \leq 1 \\ x \neq \sqrt{1-x} \end{cases} \qquad \begin{cases} x \leq 1 \\ x \leq 1 \\ x^2+x-1 \neq 0 \end{cases} \qquad \begin{cases} x \leq 1 \\ x \neq \frac{-1 \pm \sqrt{5}}{2} \end{cases} \begin{array}{l} \frac{-1-\sqrt{5}}{2} \text{ non accett.} \\ \frac{\sqrt{5}-1}{2} \text{ accett.} \end{array}$$




$$\boxed{D = \left(-\infty ; \frac{\sqrt{5}-1}{2}\right) \cup \left(\frac{\sqrt{5}-1}{2} ; 1\right]}$$

$$\boxed{y = \sqrt{\left(\frac{2}{3}\right)^{7x-x^2-1} - 1}}$$

$$\left(\frac{2}{3}\right)^{-x^2+7x-1} - 1 \geq 0$$

$$\left(\frac{2}{3}\right)^{-x^2+7x-1} \geq \left(\frac{2}{3}\right)^{0}$$

$$-x^2+7x-1 \leq 0$$

$$x^2-7x+1 \geq 0$$

$$x_{1,2} = \frac{7 \pm \sqrt{45}}{2} = \frac{7 \pm 3\sqrt{5}}{2}$$

$$x \leq \frac{7-3\sqrt{5}}{2} \quad \vee \quad x \geq \frac{7+3\sqrt{5}}{2}$$

$$\boxed{D = \left(-\infty ; \frac{7-3\sqrt{5}}{2}\right] \cup \left[\frac{7+3\sqrt{5}}{2} ; +\infty\right)}$$

$$\boxed{y = \frac{\log_a\left(5^{x^2-3x-1} - 5^{x-4}\right)}{x-5} \qquad a \in \mathbb{R}^+ - \{1\}}$$

$$\begin{cases} 5^{x^2-3x-1} - 5^{x-4} > 0 \\ x-5 \neq 0 \end{cases} \rightarrow \begin{cases} 5^{x^2-3x-1} > 5^{x-4} \\ x \neq 5 \end{cases} \rightarrow \begin{cases} x^2-3x-1 > x-4 \\ x \neq 5 \end{cases}$$

$$\begin{cases} x^2-4x+3 > 0 \\ x \neq 5 \end{cases} \rightarrow \begin{cases} (x-1)(x-3) > 0 \\ x \neq 5 \end{cases} \rightarrow \begin{cases} x < 1 \quad \vee \quad x > 3 \\ x \neq 5 \end{cases}$$

$$\boxed{D = (-\infty; 1) \cup (3; 5) \cup (5; +\infty)}$$

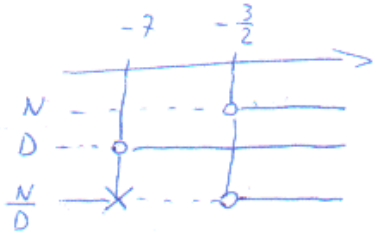
$$\boxed{y = \arcsin \frac{3x-1}{x+7}}$$

$$\begin{cases} \dfrac{3x-1}{x+7} \geq -1 \\[2ex] \dfrac{3x-1}{x+7} \leq 1 \end{cases} \rightarrow \begin{cases} \dfrac{3x-1}{x+7} + 1 \geq 0 \\[2ex] \dfrac{3x-1}{x+7} - 1 \leq 0 \end{cases} \rightarrow \begin{cases} \dfrac{3x-1+x+7}{x+7} \geq 0 \\[2ex] \dfrac{3x-1-x-7}{x+7} \leq 0 \end{cases} \rightarrow \begin{cases} \dfrac{4x+6}{x+7} \geq 0 \\[2ex] \dfrac{2x-8}{x+7} \leq 0 \end{cases}$$

prima diseq. $\dfrac{4x+6}{x+7} \geq 0 \rightarrow \dfrac{2x+3}{x+7} \geq 0$

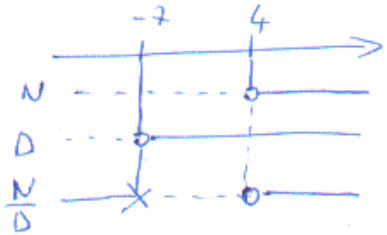
$N > 0 \rightarrow 2x+3 > 0 \rightarrow x > -\frac{3}{2}$

$D > 0 \rightarrow x+7 > 0 \rightarrow x > -7$



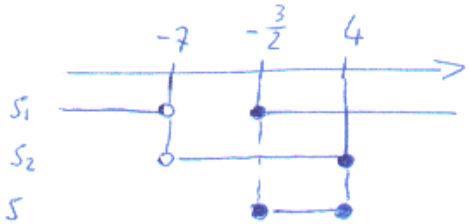

$x < -7 \quad \vee \quad x \geq -\frac{3}{2}$

seconda diseq. $\dfrac{2x-8}{x+7} \leq 0 \rightarrow \dfrac{x-4}{x+7} \leq 0$




$-7 < x \leq 4$

$$\begin{cases} x < -7 \quad \vee \quad x \geq -\frac{3}{2} \\ -7 < x \leq 4 \end{cases}$$



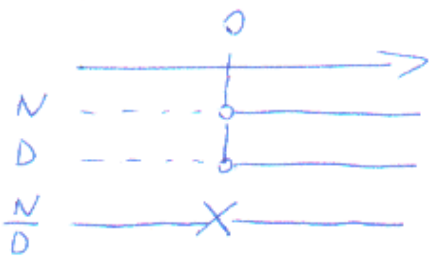

$$\boxed{D = \left[-\frac{3}{2} ; 4\right]}$$

$$\boxed{y = \frac{1}{x} \log \frac{e^x - 1}{x}}$$

$$\begin{cases} x \neq 0 \\ \dfrac{e^x - 1}{x} > 0 \end{cases}$$

$N > 0 \rightarrow e^x - 1 > 0 \rightarrow e^x > e^0 \rightarrow x > 0$

$D > 0 \rightarrow x > 0$




$$\begin{cases} x \neq 0 \\ \forall x \neq 0 \end{cases}$$

$$\boxed{D = \mathbb{R} - \{0\}}$$

$$\boxed{y = \sqrt{\frac{\log^2 x - 4}{\log x + 1}}}$$

$$\begin{cases} x > 0 \\ \dfrac{\log^2 x - 4}{\log x + 1} \geq 0 \end{cases}$$

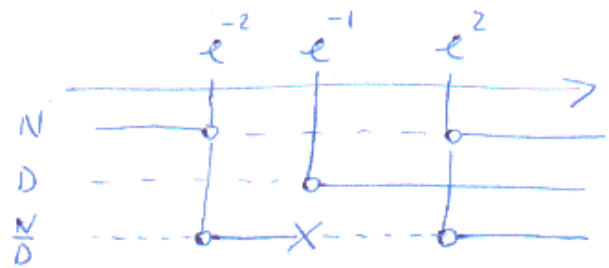
$N > 0 \rightarrow \log^2 x - 4 > 0 \rightarrow \log x < -2 \ \vee \ \log x > 2$

$x < e^{-2} \ \vee \ x > e^{2}$

$D > 0 \rightarrow \log x + 1 > 0$

$\log x > -1$

$x > e^{-1}$

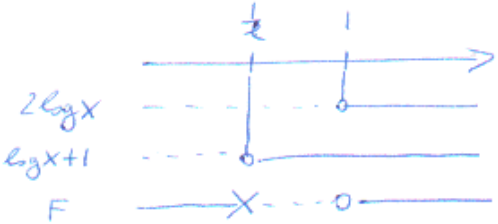


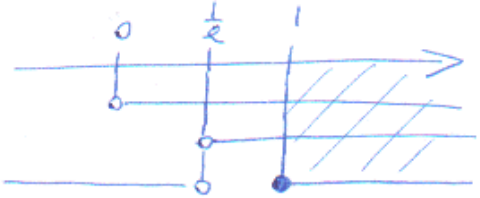

$$\begin{cases} x > 0 \\ e^{-2} \leq x < e^{-1} \ \vee \ x \geq e^{2} \end{cases}$$

$$\boxed{D = \left[\frac{1}{e^2} ; \frac{1}{e}\right) \cup \left[e^2 ; +\infty\right)}$$

$$\boxed{y = \arcsin \frac{\log x - 1}{\log x + 1}}$$

$$\begin{cases} x > 0 \\ \dfrac{\log x - 1}{\log x + 1} \leq 1 \\ \dfrac{\log x - 1}{\log x + 1} \geq -1 \end{cases} \qquad \begin{cases} x > 0 \\ \dfrac{-2}{\log x + 1} \leq 0 \\ \dfrac{2 \log x}{\log x + 1} \geq 0 \end{cases} \qquad \begin{cases} x > 0 \\ \dfrac{2}{\log x + 1} \geq 0 \\ \dfrac{2 \log x}{\log x + 1} \geq 0 \end{cases} \qquad \begin{cases} x > 0 \\ x > \frac{1}{e} \\ x < \frac{1}{e} \vee x \geq 1 \end{cases}$$



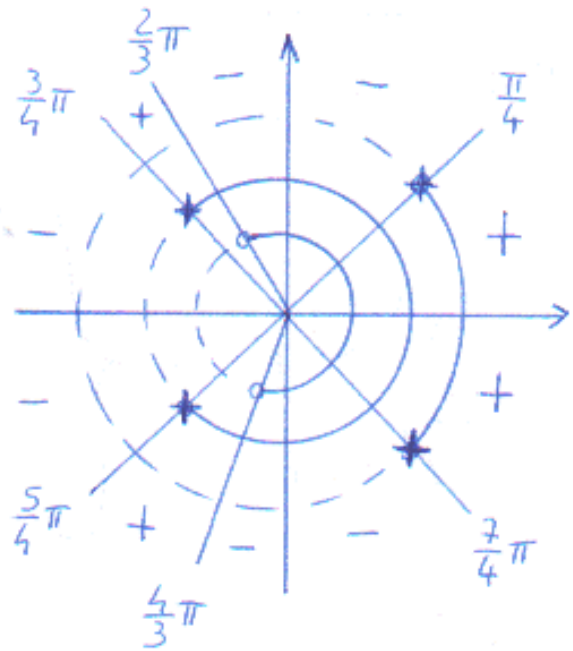




$$\boxed{D = [1; +\infty)}$$

$$y = \sqrt{\frac{2\cos x + 1}{2\cos^2 x - 1}}, \quad x \in [0; 2\pi]$$

$$\frac{2\cos x + 1}{2\cos^2 x - 1} \geq 0$$

$$\frac{2\cos x + 1}{(\sqrt{2}\cos x + 1)(\sqrt{2}\cos x - 1)} \geq 0$$




$$D = \left[0; \frac{\pi}{4}\right) \cup \left[\frac{2}{3}\pi; \frac{3}{4}\pi\right) \cup \left(\frac{5}{4}\pi; \frac{4}{3}\pi\right] \cup \left(\frac{7}{4}\pi; 2\pi\right]$$

# Definizione di limite

$$\boxed{\lim_{x \to x_0} f(x) = l}$$

$$\lim_{x \to 0^-} e^{\frac{1}{x}} = 0$$

La funzione è definita in qualsiasi intorno di 0, eccetto per x=0. Possiamo proseguire per

$$x \neq 0$$

Occorre mostrare che, comunque si scelga

$$\varepsilon > 0$$

arbitrariamente piccolo, la disuguaglianza

$$\left| e^{\frac{1}{x}} - 0 \right| < \varepsilon$$

sia verificata per tutti i valori di x di un opportuno intorno sinistro di 0. Risolviamo ora la disequazione nell'incognita x:

$$\left| e^{\frac{1}{x}} \right| < \varepsilon$$

La funzione esponenziale è sempre positiva, possiamo togliere il modulo:

$$e^{\frac{1}{x}} < \varepsilon$$

$$\ln e^{\frac{1}{x}} < \ln \varepsilon$$

$$\frac{1}{x} \ln e < \ln \varepsilon$$

$$\frac{1}{x} < \ln \varepsilon$$

$$\frac{1}{x} - \ln \varepsilon < 0$$

$$\frac{1 - x \ln \varepsilon}{x} < 0$$

$$N > 0 \rightarrow 1 - x \ln \varepsilon > 0 \rightarrow x \ln \varepsilon < 1$$

Il logaritmo di epsilon è un numero negativo, quindi:

$$N > 0 \rightarrow x \ln \varepsilon < 1 \rightarrow x > \frac{1}{\ln \varepsilon}$$

$$D > 0 \rightarrow x > 0$$

Facendo la tabella dei segni di numeratore e denominatore, si ottiene che la frazione

$$\frac{1 - x \ln \varepsilon}{x} < 0 \rightarrow x > \frac{1}{\ln \varepsilon} \wedge x < 0$$

ovvero

$$x \in \left( \frac{1}{\ln \varepsilon}; 0 \right)$$

Visto che quest'ultimo intervallo costituisce un intorno sinistro di zero, possiamo affermare che il limite è verificato.

$$\boxed{\lim_{x \to -1} \frac{x^2-2x-3}{x+1} = -4}$$

$$\lim_{x \to x_0} f(x) = \ell \iff$$

$$\forall \epsilon > 0,\ \exists \delta = \delta(\epsilon) > 0 \mid \forall x \in D \text{ con } 0 < |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \epsilon$$

$$\left| \frac{x^2-2x-3}{x+1} + 4 \right| < \epsilon \to \left| \frac{x^2-2x-3+4x+4}{x+1} \right| < \epsilon \to \left| \frac{x^2+2x+1}{x+1} \right| < \epsilon \to$$

$$\to \left| \frac{(x+1)^2}{x+1} \right| < \epsilon \to |x+1| < \epsilon \to -\epsilon < x+1 < \epsilon \to -1-\epsilon < x < -1+\epsilon$$

$$I(-1) = (-1-\epsilon\,;\, -1+\epsilon)$$

$$\boxed{\lim_{x \to 2} 2^x = 4}$$

$$\lim_{x \to x_0} f(x) = \ell \iff$$

$$\forall \epsilon > 0,\ \exists\, \delta = \delta(\epsilon) > 0 \mid \forall x \in D \text{ con } 0 < |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \epsilon$$

$$|2^x - 4| < \epsilon \rightarrow -\epsilon < 2^x - 4 < \epsilon \rightarrow 4 - \epsilon < 2^x < 4 + \epsilon \rightarrow$$

$$\log_2(4-\epsilon) < x < \log_2(4+\epsilon)$$

$$I(2) = \left(\log_2(4-\epsilon)\ ;\ \log_2(4+\epsilon)\right)$$

$$\boxed{\lim_{x \to 9} \log_3 x = 2}$$

$$\lim_{x \to x_0} f(x) = \ell \iff$$

$$\forall \epsilon > 0,\ \exists \delta = \delta(\epsilon) > 0 \mid \forall x \in D \text{ con } 0 < |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \epsilon$$

$$|\log_3 x - 2| < \epsilon \rightarrow -\epsilon < \log_3 x - 2 < \epsilon \rightarrow 2 - \epsilon < \log_3 x < 2 + \epsilon \rightarrow$$

$$\rightarrow 3^{2-\epsilon} < x < 3^{2+\epsilon} \qquad I(9) = \left(3^{2-\epsilon}; 3^{2+\epsilon}\right)$$

$$\boxed{\lim_{x \to -1} e^{x+1} = 1}$$

$$\lim_{x \to x_0} f(x) = \ell \iff$$

$$\forall \epsilon > 0, \exists \delta = \delta(\epsilon) > 0 \mid \forall x \in D \text{ con } 0 < |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \epsilon$$

$$|e^{x+1} - 1| < \epsilon \to -\epsilon < e^{x+1} - 1 < \epsilon \to 1 - \epsilon < e^{x+1} < 1 + \epsilon \to$$

$$\to \log(1-\epsilon) < x + 1 < \log(1+\epsilon) \to \log(1-\epsilon) - 1 < x < \log(1+\epsilon) - 1$$

$$I(-1) = \left(\log(1-\epsilon) - 1 \,;\, \log(1+\epsilon) - 1\right)$$

$$\boxed{\lim_{x\to\infty} f(x) = l}$$

$$\boxed{\lim_{x\to+\infty} \frac{e^x+1}{e^x} = 1}$$

La funzione è definita per ogni x reale. Occorre mostrare che, comunque si scelga

$$\varepsilon > 0$$

arbitrariamente piccolo, la disuguaglianza

$$\left|\frac{e^x+1}{e^x} - 1\right| < \varepsilon$$

sia verificata per tutti i valori di x di un opportuno intorno di +infinito. Risolviamo ora la disequazione nell'incognita x:

$$\left|\frac{e^x+1}{e^x} - 1\right| < \varepsilon \to \left|\frac{e^x+1-e^x}{e^x}\right| < \varepsilon$$

$$\left|\frac{1}{e^x}\right| < \varepsilon \to \frac{1}{|e^x|} < \varepsilon \to |e^x| > \frac{1}{\varepsilon}$$

$$e^x < -\frac{1}{\varepsilon} \vee e^x > +\frac{1}{\varepsilon}$$

Analizzando gli intervalli:

$$e^x < -\frac{1}{\varepsilon}$$

risulta impossibile, mentre

$$e^x > +\frac{1}{\varepsilon} \rightarrow x > \ln\frac{1}{\varepsilon} \rightarrow x > -\ln\varepsilon$$

Otteniamo quindi

$$(-\ln\varepsilon; +\infty)$$

Visto che quest'ultimo intervallo costituisce un intorno di +infinito, possiamo affermare che il limite è verificato.

$$\boxed{\lim_{x \to \infty} \frac{9x-3}{3x} = 3}$$

$$\lim_{x \to \infty} f(x) = \ell \iff$$

$$\forall \epsilon > 0, \ \exists N = N(\epsilon) > 0 \mid \forall x \in D \text{ con } |x| > N \Rightarrow |f(x) - \ell| < \epsilon$$

$$|f(x) - \ell| < \epsilon \to \left| \frac{9x-3}{3x} - 3 \right| < \epsilon \to \left| \frac{\cancel{9x} - 3 - \cancel{9x}}{3x} \right| < \epsilon \to \frac{1}{|x|} < \epsilon \to$$

$$\to |x| > \frac{1}{\epsilon} \to x < -\frac{1}{\epsilon} \vee x > \frac{1}{\epsilon}$$

$$I(\infty) = \left(-\infty; -\frac{1}{\epsilon}\right) \cup \left(\frac{1}{\epsilon}; +\infty\right)$$

$$\boxed{\lim_{x\to\infty} \log \frac{x+1}{x} = 0}$$

$$\lim_{x\to\infty} f(x) = \ell \iff$$

$$\forall \epsilon>0,\ \exists N=N(\epsilon)>0 \mid \forall x\in D \text{ con } |x|>N \Rightarrow |f(x)-\ell|<\epsilon$$

$$\left|\log \frac{x+1}{x}\right| < \epsilon \to -\epsilon < \log\frac{x+1}{x} < \epsilon \to$$

$$\to e^{-\epsilon} < \frac{x+1}{x} < e^{\epsilon} \to e^{-\epsilon} < 1+\frac{1}{x} < e^{\epsilon} \to e^{-\epsilon}-1 < \frac{1}{x} < e^{\epsilon}-1$$

se $x>0$ $\quad e^{-\epsilon}-1 < \frac{1}{x} < e^{\epsilon}-1 \to \frac{1}{x} < e^{\epsilon}-1 \to x > \frac{1}{e^{\epsilon}-1}$

se $x<0$ $\quad e^{-\epsilon}-1 < \frac{1}{x} < e^{\epsilon}-1 \to \frac{1}{x} > e^{-\epsilon}-1 \to 1 < x(e^{-\epsilon}-1) \to \frac{1}{e^{-\epsilon}-1} > x \to$

$$\to x < \frac{1}{e^{-\epsilon}-1}$$

$$x < \frac{1}{e^{-\epsilon}-1} \vee x > \frac{1}{e^{\epsilon}-1} \qquad I(\infty) = \left(-\infty; \frac{1}{e^{-\epsilon}-1}\right) \cup \left(\frac{1}{e^{\epsilon}-1}; +\infty\right)$$

$$\boxed{\lim_{x \to -\infty} 3^{2x} = 0}$$

$$\lim_{x \to -\infty} f(x) = \ell \iff$$

$$\forall \epsilon > 0, \ \exists N = N(\epsilon) > 0 \ / \ \forall x \in D \text{ con } x < -N \Rightarrow |f(x) - \ell| < \epsilon$$

$$|3^{2x}| < \epsilon \to 3^{2x} < \epsilon \to 2x < \log_3 \epsilon \to x < \frac{1}{2} \log_3 \epsilon$$

$$I(-\infty) = \left(-\infty ; \frac{1}{2} \log_3 \epsilon\right)$$

$$\boxed{\lim_{x \to x_0} f(x) = \infty}$$

$$\boxed{\lim_{x \to 2} \frac{2}{5x - 10} = \infty}$$

La funzione è definita per

$$x \neq 2$$

Occorre mostrare che, comunque si scelga

$$M > 0$$

arbitrariamente grande, la disuguaglianza

$$\left| \frac{2}{5x - 10} \right| > M$$

sia verificata per tutti i valori di x di un opportuno intorno di 2.

Risolviamo ora la disequazione nell'incognita x:

$$\left|\frac{2}{5x-10}\right| > M$$

$$\frac{2}{|5x-10|} > M$$

$$|5x-10| < \frac{2}{M}$$

$$5x-10 > -\frac{2}{M} \ \wedge \ 5x-10 < +\frac{2}{M}$$

$$5x > 10-\frac{2}{M} \ \wedge \ 5x < 10+\frac{2}{M}$$

$$x > 2-\frac{2}{5M} \ \wedge \ x < 2+\frac{2}{5M}$$

$$\boxed{\left(2-\frac{2}{5M};2+\frac{2}{5M}\right)}$$

Visto che quest'ultimo intervallo costituisce un intorno di 2, possiamo affermare che il limite è verificato.

$$\boxed{\lim_{x \to -1} \frac{2}{x+1} = \infty}$$

$$\lim_{x \to x_0} f(x) = \infty \iff$$

$$\forall M > 0,\ \exists\, \delta = \delta(M) > 0 \mid \forall x \in D \text{ con } 0 < |x - x_0| < \delta \Rightarrow |f(x)| > M$$

$$\left|\frac{2}{x+1}\right| > M \rightarrow \frac{2}{|x+1|} > M \rightarrow |x+1| < \frac{2}{M} \rightarrow$$

$$\rightarrow -\frac{2}{M} < x+1 < \frac{2}{M} \rightarrow -1 - \frac{2}{M} < x < -1 + \frac{2}{M}$$

$$I(-1) = \left(-1 - \frac{2}{M}\ ;\ -1 + \frac{2}{M}\right)$$

$$\boxed{\lim_{x \to 0} \frac{1}{e^x - 1} = \infty}$$

$$\lim_{x \to x_0} f(x) = \infty \iff$$

$$\forall M > 0, \exists \delta = \delta(M) > 0 \mid \forall x \in D \text{ con } 0 < |x - x_0| < \delta \Rightarrow |f(x)| > M$$

$$\left|\frac{1}{e^x - 1}\right| > M \to \frac{1}{|e^x - 1|} > M \to |e^x - 1| < \frac{1}{M} \to -\frac{1}{M} < e^x - 1 < \frac{1}{M} \to$$

$$\to 1 - \frac{1}{M} < e^x < 1 + \frac{1}{M} \to \log\left(1 - \frac{1}{M}\right) < x < \log\left(1 + \frac{1}{M}\right)$$

$$I(0) = \left(\log\left(1 - \frac{1}{M}\right); \log\left(1 + \frac{1}{M}\right)\right)$$

$$\boxed{\lim_{x \to 0} \log_3 x^2 = \infty}$$

$$\lim_{x \to x_0} f(x) = \infty \iff$$

$$\forall M > 0,\ \exists \delta = \delta(M) > 0 \mid \forall x \in D \text{ con } 0 < |x - x_0| < \delta \Rightarrow |f(x)| > M$$

$$|\log_3 x^2| > M \rightarrow -2 \log_3 |x| > M \rightarrow \log_3 |x| < -\frac{M}{2} \rightarrow$$

$$\rightarrow |x| < 3^{-\frac{M}{2}} \rightarrow -3^{-\frac{M}{2}} < x < 3^{-\frac{M}{2}}$$

$$I(0) = \left(-3^{-\frac{M}{2}};\ 3^{-\frac{M}{2}}\right)$$

$$\boxed{\lim_{x\to\infty} f(x) = \infty}$$

$$\boxed{\lim_{x\to\infty} \left(2x^3-1\right) = \infty}$$

$$\forall M>0,\ \exists N=N(M)>0 \mid \forall x\in D \text{ con } |x|>N \Rightarrow |f(x)|>M$$

$$|2x^3-1|>M \rightarrow \quad 2x^3-1<-M \quad \vee \quad 2x^3-1>M \rightarrow$$

$$\rightarrow \quad 2x^3<1-M \quad \vee \quad 2x^3>M+1 \rightarrow$$

$$\rightarrow \quad x^3<-\frac{M-1}{2} \quad \vee \quad x^3>\frac{M+1}{2} \rightarrow$$

$$\rightarrow \quad x<-\sqrt[3]{\frac{M-1}{2}} \quad \vee \quad x>\sqrt[3]{\frac{M+1}{2}}$$

$$\boxed{I(\infty) = \left(-\infty\,;-\sqrt[3]{\frac{M-1}{2}}\right) \cup \left(\sqrt[3]{\frac{M+1}{2}}\,;+\infty\right)}$$

$$\boxed{\lim_{x \to \infty} \log(1+x^2) = \infty}$$

$$\lim_{x \to +\infty} f(x) = +\infty \iff$$

$$\forall M > 0, \exists N = N(M) > 0 \mid \forall x \in D \text{ con } x > N \Rightarrow f(x) > M$$

$$1 + e^{2x} > M \to e^{2x} > M - 1 \to 2x > \log(M-1) \to$$

$$\to x > \frac{1}{2} \log(M-1) \qquad I(+\infty) = \left(\frac{1}{2}\log(M-1) ; +\infty\right)$$

$$\boxed{\lim_{x \to +\infty} \left(1+e^{2x}\right) = +\infty}$$

$$\lim_{x \to +\infty} f(x) = +\infty \iff$$

$$\forall M>0,\ \exists N=N(M)>0 \mid \forall x \in D \text{ con } x>N \Rightarrow f(x)>M$$

$$1+e^{2x}>M \to e^{2x}>M-1 \to 2x>\log(M-1) \to$$

$$\to x>\frac{1}{2}\log(M-1) \qquad I(+\infty)=\left(\frac{1}{2}\log(M-1)\,;\,+\infty\right)$$

# Calcolo dei limiti

$$\boxed{\lim_{x\to 0} \frac{e^{2x}-1}{3x}}$$

$$\boxed{\lim_{x\to 0} \frac{e^{x}-1}{x} = 1}$$

$$\lim_{x\to 0} \frac{\frac{e^{2x}-1}{2x}\cdot 2x}{3x} \Rightarrow \lim_{x\to 0} \frac{(e^{2x}-1)\cdot \frac{2x}{2x}}{3x} \Rightarrow \lim_{x\to 0} \frac{\frac{e^{2x}-1}{2x}\cdot 2x}{3x} = \lim_{x\to 0} \frac{e^{2x}-1}{2x}\cdot \lim_{x\to 0} \frac{2x}{3x}$$

In definitiva: $$\boxed{\lim_{x\to 0} \frac{e^{2x}-1}{3x} = \frac{2}{3}}$$

**Procedimento alternativo**: moltiplichiamo e dividiamo il denominatore per due

$$\lim_{x\to 0} \frac{e^{2x}-1}{2x\cdot \frac{3}{2}} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$

$$\boxed{\lim_{x\to\infty}\left(1+\frac{1}{2x}\right)^{3x}}$$

Per poter usare il limite notevole: $\boxed{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=1}$

moltiplichiamo e dividiamo l'esponente per 2:

$$\lim_{x\to\infty}\left(1+\frac{1}{2x}\right)^{\frac{2}{2}3x}$$

Usando la proprietà delle potenze, si ottiene:

$$\lim_{x\to\infty}\left[\left(1+\frac{1}{2x}\right)^{2x}\right]^{\frac{3}{2}} \quad \Rightarrow \quad \lim_{x\to\infty}\left[\left(1+\frac{1}{2x}\right)^{2x}\right]^{\frac{3}{2}}=e^{\frac{3}{2}}$$

$$\boxed{\lim_{x\to 0} \frac{1-\cos(3x)}{x^2}}$$

Moltiplichiamo e dividiamo per 9:

$$\lim_{x\to 0} \frac{9(1-\cos(3x))}{9x^2} = 9 \lim_{x\to 0} \frac{1-\cos(3x)}{(3x)^2}$$

Poniamo $3x = t$ e osserviamo che quando x tende a zero, t tende a zero, in questo modo il limite si riscrive come:

$$9 \lim_{x\to 0} \frac{1-\cos(3x)}{(3x)^2} =$$

$$9 \overbrace{\lim_{t\to 0} \frac{1-\cos(t)}{t^2}}^{\text{limite notevole}} = 9 \cdot \frac{1}{2} = \frac{9}{2}$$

$$\boxed{\lim_{x\to 1} \frac{arctg(x-1)}{2(1-x)}}$$

Inizierei col porre:

$x - 1 = t$

da cui

$1 - x = -t$

Ora, per

$x \to 1, t \to 0$

Quindi sostituendo nel tuo limite hai:

$$\lim_{t\to 0} \frac{arctg(t)}{2(-t)} = -\frac{1}{2} \lim_{t\to 0} \frac{arctg(t)}{(t)} = -\frac{1}{2}$$

dove, nell'ultimo passaggio abbiamo usato il limite notevole:

$$\lim_{t\to 0} \frac{arctg(t)}{t} = 1$$

$$\boxed{\lim_{x\to 0} \frac{1-e^{2x}}{\sin(3x)}}$$

ci torneranno utili i limiti notevoli:

$$\lim_{t\to 0} \frac{\sin(t)}{t} = 1$$

$$\lim_{t\to 0} \frac{e^t - 1}{t} = 1$$

$$\lim_{x\to 0} (\ 1 - e^{2x}) \frac{1}{\sin(3x)}$$

A questo punto moltiplichiamo e dividiamo per $2x$ ottenendo:

$$\lim_{x\to 0} \frac{1-e^{2x}}{2x}\ 2x\ \frac{1}{\sin(3x)}$$

In questo modo ci siamo ricondotti quasi al limite notevole del esponenziale (a meno di un segno)

Per ricondurci anche al limite notevole del seno moltiplichiamo e dividiamo per $3x$

$$\lim_{x\to 0} \frac{1-e^{2x}}{2x} \frac{2x}{3x} \frac{3x}{\sin(3x)}$$

Semplificando in modo opportuno abbiamo:

$$\lim_{x\to 0} \frac{1-e^{2x}}{2x} \frac{2}{3} \frac{3x}{\sin(3x)}$$

dunque:

$$\frac{2}{3} \lim_{x\to 0} \frac{1-e^{2x}}{2x} \frac{3x}{\sin(3x)}$$

Spezziamo il limite come prodotto di più limiti:

$$\frac{2}{3}\lim_{x\to 0}\frac{1-e^{2x}}{2x}\lim_{x\to 0}\frac{3x}{\sin(3x)}$$

Risolviamoli separatamente:

$$lim_{x\to 0}\frac{1-e^{2x}}{2x}=-\lim_{x\to 0}\frac{e^{2x}-1}{2x}=-1$$

mentre:

$$\lim_{x\to 0}\frac{3x}{\sin(3x)}=1$$

perché è il reciproco del limite notevole del seno. Quindi:

$$\frac{2}{3}\overbrace{\lim_{x\to 0}\frac{1-e^{2x}}{2x}}^{=-1}\overbrace{\lim_{x\to 0}\frac{3x}{\sin(3x)}}^{=1}=-\frac{2}{3}$$

$$\boxed{\lim_{x\to 0}\frac{e^{x}-e^{-x}}{senx}}$$

Sostituendo 0 al posto di $x$ si ha la forma $\frac{0}{0}$.

Ricordiamo i limiti notevoli

$\lim_{x\to 0}\frac{senx}{x}=1$ e $\lim_{x\to 0}\frac{e^{x}-1}{x}=1$ per cui dividiamo al numeratore e denominatore per $x$ e al numeratore aggiungiamo e sottraiamo 1:

$$\lim_{x\to 0}\frac{\frac{e^{x}-1-(e^{-x}0-1)}{x}}{\frac{senx}{x}}=\lim_{x\to 0}\frac{\left(\frac{e^{x}-1}{x}\right)+\left(\frac{e^{-x}-1}{-x}\right)}{\frac{senx}{x}}=2$$

$$\boxed{\lim_{x\to 1}\frac{e^x - e}{x-1}}$$

Posto $y = x - 1 \quad \rightarrow \quad x = y + 1$ si ha:

$$\lim_{y\to 0}\frac{e^{y+1} - e}{y+1-1} = \lim_{y\to 0}\frac{e^y \cdot e - e}{y} = \lim_{y\to 0}\frac{e(e^y - 1)}{y} = e$$

$$\boxed{\lim_{x\to 0}\frac{e^{x^2}-cosx}{x^2}}$$

Ricordiamo i limiti notevoli

$$\lim_{x\to 0}\frac{e^{x}-1}{x}=1 \qquad \lim_{x\to 0}\frac{senx}{x}=1 \qquad \lim_{x\to 0}\frac{1-cosx}{x^2}=\frac{1}{2}$$

Aggiungiamo e sottraiamo 1 al numeratore, si ha:

$$\lim_{x\to 0}\frac{e^{x^2}-1}{x^2}-\frac{(cosx-1)}{x^2}=1+\frac{1}{2}=\frac{3}{2}$$

$$\boxed{\lim_{x \to 0} \frac{sen\ 2x}{ln(1+x)}}$$

$$\lim_{x \to 0} \frac{sen\ 2x}{ln(1+x)} = \lim_{x \to 0} \frac{\frac{sen2x}{2x} \cdot 2x}{\frac{ln(1+x)}{x} \cdot x} = 2$$

$$\boxed{\lim_{x\to+\infty}\left(\frac{x+8}{x-2}\right)^{x} = 1^{\infty}}$$

Ricorda il limite notevole $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e$

Mettiamo in evidenza $x$ al nominatore e al denominatore:

$$\lim_{x\to+\infty}\left(\frac{x\left(1+\frac{8}{x}\right)}{x\left(1+\frac{(-2)}{x}\right)}\right)^{x} = \lim_{x\to+\infty}\frac{\left(1+\frac{8}{x}\right)^{x}}{\left(1+\frac{(-2)}{x}\right)^{x}} = \frac{e^{8}}{e^{-2}} = e^{10}$$

$$\lim_{x\to\infty}\left(1+\frac{n}{x}\right)^{x}=e^{n} \qquad \lim_{x\to\infty}\left(1+\frac{n}{x}\right)^{mx}=e^{mn} \qquad \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e \qquad \lim_{x\to\infty}\left(1-\frac{1}{x}\right)^{x}=\frac{1}{e}$$

$$\lim_{x\to\infty}\left(1+\frac{1}{2x}\right)^{2x}=e$$

$$\lim_{x\to\infty}\left(1+\frac{1}{2x}\right)^{x}=\lim_{x\to\infty}\left[\left(1+\frac{1}{2x}\right)^{2x}\right]^{\frac{1}{2}}=e^{\frac{1}{2}}=\sqrt{e}$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{2x}=\lim_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^{x}\right]^{2}=e^{2}$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x}=\lim_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^{x}\right]^{3}=e^{3}$$

$$\lim_{x\to\infty}\left(1-\frac{2}{x}\right)^{x} = \lim_{x\to\infty}\left[\left(1+\frac{1}{\left(-\frac{1}{2}x\right)}\right)^{-\frac{1}{2}x}\right]^{-2} = e^{-2} = \frac{1}{e^{2}}$$

$$\lim_{x\to\infty}\left(1-\frac{3}{2x}\right)^{x} = \lim_{x\to\infty}\left[\left(1+\frac{1}{\left(-\frac{2}{3}x\right)}\right)^{-\frac{2}{3}x}\right]^{-\frac{3}{2}} = e^{-\frac{3}{2}}$$

$$\lim_{x\to 0}(1+nx)^{\frac{1}{x}} = e^{n} \qquad \lim_{x\to 0}(1+x)^{\frac{1}{x}} = e \qquad \lim_{x\to 0}(1-x)^{\frac{1}{x}} = \frac{1}{e}$$

$$\lim_{x\to 0}(1+2x)^{\frac{1}{x}} = \lim_{x\to 0}\left[(1+2x)^{\frac{1}{2x}}\right]^{2} = e^{2}$$

$$\lim_{x\to 0}(1+x)^{\frac{1}{3x}} = \lim_{x\to 0}\left[(1+x)^{\frac{1}{x}}\right]^{\frac{1}{3}} = e^{\frac{1}{3}}$$

$$\lim_{x\to 0}\frac{a^x-1}{x}=\ln a \qquad \lim_{x\to 0}\frac{e^x-1}{x}=1$$

$$\lim_{x\to 3}\frac{2^{x-3}-1}{x-3}$$

$t=x-3$ se $x\to 3$ allora $t\to 0$

$$\lim_{x\to 3}\frac{2^{x-3}-1}{x-3}=\lim_{t\to 0}\frac{2^t-1}{t}=\log 2$$

$$\lim_{x\to 0}\frac{e^{2x}-1}{x}=\lim_{x\to 0}\left[\frac{e^{2x}-1}{2x}\cdot 2\right]=2$$

$$\lim_{x\to -2}\frac{e^{x+2}-1}{x+2}$$

$t=x+2$ se $x\to -2$ allora $t\to 0$

$$\lim_{x\to -2}\frac{e^{x+2}-1}{x+2}=\lim_{t\to 0}\frac{e^t-1}{t}=1$$

$$\lim_{x\to 0}\frac{2^{2x}-1}{3^x-1}=\lim_{x\to 0}\frac{\cancel{2x}\cdot\frac{2^{2x}-1}{2x}}{\cancel{x}\cdot\frac{3^x-1}{x}}=\frac{2\log 2}{\log 3}$$

$$\lim_{x\to 1} \frac{e^{x-1}-1}{1-\cos(1-x)} = \lim_{t\to 0} \frac{e^{t}-1}{1-\cos(-t)} = \lim_{t\to 0} \frac{\cancel{t}\cdot\frac{e^{t}-1}{t}}{t^{\cancel{2}}\cdot\frac{1-\cos t}{t^2}} = \infty$$

$t = x-1$

se $x \to 1$

allora $t \to 0$

$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=1 \qquad \lim_{x\to 0}\frac{\log_a(1+x)}{x}=\log_a e$$

$$\lim_{x\to 0}\frac{\log_3(1+x)}{2x}=\lim_{x\to 0}\left[\frac{1}{2}\cdot\frac{\log_3(1+x)}{x}\right]=\frac{1}{2}\log_3 e$$

$$\lim_{x\to 0}\frac{\log(1+3x)}{x}=\lim_{x\to 0}\left[3\,\frac{\log(1+3x)}{3x}\right]=3$$

$$\lim_{x\to 0}\frac{\log(1+2x)}{\log(1+x)}=\lim_{x\to 0}\frac{2\cancel{x}\cdot\dfrac{\log(1+2x)}{2x}}{\cancel{x}\cdot\dfrac{\log(1+x)}{x}}=2$$

$$\lim_{x\to 0}\frac{\log_2(1+x)}{3^x-1}=\lim_{x\to 0}\frac{\dfrac{\log_2(1+x)}{x}}{\dfrac{3^x-1}{x}}=\frac{\log_2 e}{\log 3}=\frac{1}{\log 2\,\log 3}$$

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \lim_{x\to 0}\frac{tgx}{x}=1$$

$$\lim_{x\to 0}\frac{\sin 3x}{3x}=1$$

$$\lim_{x\to 0}\frac{\sin 4x}{x}=\lim_{x\to 0}\left(4\cdot\frac{\sin 4x}{4x}\right)=4$$

$$\lim_{x\to 0}\frac{\sin\frac{x}{2}}{x}=\lim_{x\to 0}\left(\frac{1}{2}\cdot\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)=\frac{1}{2}$$

$$\lim_{x\to 0}\frac{tg\,x}{x}=\lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\frac{1}{\cos x}\right)=1$$

$$\lim_{x\to 0}\frac{\sin 2x}{x\cos x}=\lim_{x\to 0}\frac{2\sin x\cancel{\cos x}}{x\cancel{\cos x}}=\lim_{x\to 0}\left(2\cdot\frac{\sin x}{x}\right)=2$$

$$\lim_{x\to 0} \frac{\operatorname{tg} x}{3x} = \lim_{x\to 0}\left(\frac{1}{3\cos x}\cdot\frac{\sin x}{x}\right) = \frac{1}{3}$$

$$\lim_{x\to 0} \frac{\sin^2 x}{x} = \lim_{x\to 0}\left(\sin x\cdot\frac{\sin x}{x}\right) = 0$$

$$\lim_{x\to 0} \frac{\operatorname{tg} x + \sin x}{3x} = \lim_{x\to 0}\left[\frac{1}{3\cos x}\cdot\frac{\sin x}{x} + \frac{1}{3}\cdot\frac{\sin x}{x}\right] = \frac{1}{3}+\frac{1}{3} = \frac{2}{3}$$

$$\lim_{x\to 0} \frac{3\sin x + 2x}{x} = \lim_{x\to 0}\left[3\,\frac{\sin x}{x} + 2\right] = 3+2 = 5$$

$$\lim_{x\to 0} \frac{2x + 3\sin x}{5\operatorname{tg} x} = \lim_{x\to 0} \frac{2 + 3\cdot\frac{\sin x}{x}}{\frac{5}{\cos x}\cdot\frac{\sin x}{x}} = \frac{2+3}{5} = 1$$

$$\lim_{x\to 0} \frac{\sin 3x}{\sin 5x} = \lim_{x\to 0} \frac{\cancel{3x}\cdot\frac{\sin 3x}{3x}}{\cancel{5x}\cdot\frac{\sin 5x}{5x}} = \frac{3}{5}$$

$$\lim_{x\to 0}\frac{1-\cos x}{x}=0 \qquad \lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}$$

$$\lim_{x\to 0}\frac{x^2}{1-\cos x}=\lim_{x\to 0}\frac{1}{\frac{1-\cos x}{x^2}}=\frac{1}{\frac{1}{2}}=2$$

$$\lim_{x\to 0}\frac{\sin^2 x}{1-\cos x}=\lim_{x\to 0}\frac{\left(\frac{\sin x}{x}\right)^2}{\frac{1-\cos x}{x^2}}=\frac{1}{\frac{1}{2}}=2$$

$$\lim_{x\to 0^+}\frac{\sqrt{1-\cos x}}{x}=\lim_{x\to 0^+}\left(\frac{\sqrt{1-\cos x}}{x}\cdot\frac{\sqrt{1+\cos x}}{\sqrt{1+\cos x}}\right)=\lim_{x\to 0^+}\frac{\sqrt{1-\cos^2 x}}{x\sqrt{1+\cos x}}=$$

$$=\lim_{x\to 0^+}\left(\frac{\sin x}{x}\cdot\frac{1}{\sqrt{1+\cos x}}\right)=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$$

$$\lim_{x\to 0} \frac{\sin^2 x}{1-\cos^3 x} = \lim_{x\to 0} \frac{\sin^2 x}{(1-\cos x)(1+\cos x+\cos^2 x)} =$$

$$\lim_{x\to 0} \frac{\frac{\sin^2 x}{x^2}}{\frac{1-\cos x}{x^2}\cdot(1+\cos x+\cos^2 x)} = \frac{1}{\frac{1}{2}\cdot 3} = \frac{2}{3}$$

$$\lim_{x\to 0} \frac{1-\cos^3 x}{x\sin x} = \lim_{x\to 0} \frac{(1-\cos x)(1+\cos x+\cos^2 x)}{x\sin x} =$$

$$\lim_{x\to 0} \frac{\frac{1-\cos x}{x^2}\cdot(1+\cos x+\cos^2 x)}{\frac{\sin x}{x}} = \frac{\frac{1}{2}\cdot 3}{1} = \frac{3}{2}$$

# Forme indeterminate

$$\lim_{x\to 1} \frac{x^2-1}{x^2-3x+2} = \lim_{x\to 1} \frac{(x+1)\cancel{(x-1)}}{(x-2)\cancel{(x-1)}} = -2$$

$$\lim_{x\to 2} \frac{x^2-4x+4}{x^2-4} = \lim_{x\to 2} \frac{(x-2)^{\cancel{2}}}{(x+2)\cancel{(x-2)}} = 0$$

$$\lim_{x\to \frac{1}{2}} \frac{8x^3+2x^2-5x+1}{8x^3+10x^2-11x+2} = \lim_{x\to \frac{1}{2}} \frac{\cancel{\left(x-\frac{1}{2}\right)}(8x^2+6x-2)}{\cancel{\left(x-\frac{1}{2}\right)}(8x^2+14x-4)} = \frac{3}{5}$$

| | | | | |
|---|---|---|---|---|
| | 8 | 2 | -5 | 1 |
| $\frac{1}{2}$ | | 4 | 3 | -1 |
| | 8 | 6 | -2 | 0 |

$$\lim_{x\to 2} \frac{16-x^4}{2-x} = \lim_{x\to 2} \frac{(4+x^2)(2+x)\cancel{(2-x)}}{\cancel{(2-x)}} = 32$$

| | | | | |
|---|---|---|---|---|
| | 8 | 10 | -11 | 2 |
| $\frac{1}{2}$ | | 4 | 7 | -2 |
| | 8 | 14 | -4 | 0 |

$$\lim_{x\to\infty} \frac{3x^2+5x-1}{4x^2-5x+1} = \lim_{x\to\infty} \frac{3\cancel{x^2}}{4\cancel{x^2}} = \frac{3}{4}$$

$$\lim_{x\to+\infty} \frac{x^2-x}{3x+4} = \lim_{x\to+\infty} \frac{x^{\cancel{2}}}{3\cancel{x}} = +\infty$$

$$\lim_{x\to\infty} \frac{2}{x^2+1} = 0$$

$$\lim_{x\to-\infty} \frac{x+1}{3x^2-4} = \lim_{x\to-\infty} \frac{\cancel{x}}{3x^{\cancel{2}}} = 0$$

$$\lim_{x\to1} \frac{\sqrt{x}-1}{x-1} = \lim_{x\to1} \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{(x-1)(\sqrt{x}+1)} = \lim_{x\to1} \frac{\cancel{x-1}}{\cancel{(x-1)}(\sqrt{x}+1)} = \frac{1}{2}$$

$$\lim_{x\to2} \frac{\sqrt{2}-\sqrt{x}}{2-x} = \lim_{x\to2} \frac{(\sqrt{2}-\sqrt{x})(\sqrt{2}+\sqrt{x})}{(2-x)(\sqrt{2}+\sqrt{x})} = \lim_{x\to2} \frac{\cancel{2-x}}{\cancel{(2-x)}(\sqrt{2}+\sqrt{x})} = \frac{1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{4}$$

$$\lim_{x\to+\infty} (x+1)^{-\frac{1}{\log x}}$$

forma indeterminata $[\infty^0]$

$$\lim_{x\to+\infty} (x+1)^{-\frac{1}{\log x}} = \lim_{x\to+\infty} e^{-\frac{\log(x+1)}{\log x}} = e^{\lim_{x\to+\infty}\left(-\frac{\log x + \log\left(1+\frac{1}{x}\right)}{\log x}\right)} =$$

$$= e^{\lim_{x\to+\infty}\left[-\frac{\cancel{\log x}\left[1+\frac{\log\left(1+\frac{1}{x}\right)}{\log x}\right]}{\cancel{\log x}}\right]} = e^{-1} = \frac{1}{e}$$

$$\lim_{x\to+\infty} \left(\frac{x-3}{x+4}\right)^{\frac{x^2-1}{2x}}$$

forma indeterminata $[1^\infty]$

$$\frac{x-3}{x+4} = 1 + \frac{1}{f(x)} \to \frac{x-3}{x+4} - 1 = \frac{1}{f(x)} \to \frac{x-3-x-4}{x+4} = \frac{1}{f(x)} \to$$

$$\to \frac{1}{f(x)} = \frac{-7}{x+4} \to f(x) = -\frac{x+4}{7}$$

$$\lim_{x\to+\infty} \left(\frac{x-3}{x+4}\right)^{\frac{x^2-1}{2x}} = \lim_{x\to+\infty}\left[\left(1+\frac{1}{-\frac{x+4}{7}}\right)^{-\frac{x+4}{7}}\right]^{-\frac{7}{x+4}\cdot\frac{x^2-1}{2x}} = e^{\lim_{x\to+\infty}\frac{-7x^2+7}{2x^2+8x}} =$$

$$= e^{\lim_{x\to+\infty}\frac{\cancel{x^2}\left(-7+\frac{7}{x^2}\right)}{\cancel{x^2}\left(2+\frac{8}{x}\right)}} = e^{-\frac{7}{2}}$$

$$\lim_{x\to 0^+} x^{\frac{1}{\log^2 x}}$$ forma indeterminata $[0^0]$

$$\lim_{x\to 0^+} x^{\frac{1}{\log^2 x}} = \lim_{x\to 0^+} e^{\frac{\log x}{\log^2 x}} = e^{\lim_{x\to 0^+} \frac{1}{\log x}} = e^0 = 1$$

$$\lim_{x\to 0^+} x^{\frac{2}{\log x}}$$ forma indeterminata $[0^0]$

$$\lim_{x\to 0^+} x^{\frac{2}{\log x}} = \lim_{x\to 0^+} e^{\frac{2}{\cancel{\log x}}\cdot \cancel{\log x}} = e^2$$

# Limiti immediati particolari

$$\lim_{x \to \left(\frac{\pi}{3}\right)^+} e^{\frac{1}{2\cos x - 1}} = e^{\frac{1}{0^-}} = e^{-\infty} = 0$$

$$\lim_{x \to \left(\frac{\pi}{6}\right)^-} e^{\frac{1}{1 - 2\sin x}} = e^{\frac{1}{0^+}} = e^{+\infty} = +\infty$$

$$\lim_{x \to \left(\frac{\pi}{4}\right)^-} e^{\frac{1}{\operatorname{tg} x - 1}} = e^{\frac{1}{0^-}} = e^{-\infty} = 0$$

$$\lim_{x \to \left(\frac{\pi}{4}\right)^+} e^{\frac{1}{\operatorname{tg} x - 1}} = e^{\frac{1}{0^+}} = e^{+\infty} = +\infty$$

$$\lim_{x \to +\infty} \frac{3e^x - 5}{1 + 4e^x} = \lim_{x \to +\infty} \frac{\cancel{e^x}\left(3 - \frac{5}{e^x}\right)}{\cancel{e^x}\left(\frac{1}{e^x} + 4\right)} = \frac{3}{4}$$

$$\lim_{x \to -\infty} \frac{3e^x - 5}{1 + 4e^x} = -5$$

$$\lim_{x \to 1^-} \left(\frac{1}{2}\right)^{\frac{1-3x}{1-x}} = \left(\frac{1}{2}\right)^{-\infty} = +\infty$$

$$\lim_{x \to 1^+} \left(\frac{1}{2}\right)^{\frac{1-3x}{1-x}} = \left(\frac{1}{2}\right)^{+\infty} = 0$$

# Infinitesimi e infiniti

$f(x) = x^2 - 1$ per $x \to 1$

$$\lim_{x \to 1} \frac{f(x)}{(x-1)^\alpha} = \lim_{x \to 1} \frac{x^2-1}{(x-1)^\alpha} = \lim_{x \to 1} \frac{(x+1)(x-1)}{(x-1)^\alpha} = 2 \quad \text{se } \alpha = 1$$

$f(x)$ è un infinitesimo del 1° ordine per $x \to 1$

$f(x) \sim 2(x-1)$ parte principale

$f(x) = x^3 - 5x^2$ per $x \to 0$

$$\lim_{x \to 0} \frac{f(x)}{x^\alpha} = \lim_{x \to 0} \frac{x^2(x-5)}{x^\alpha} = -5 \quad \text{se } \alpha = 2$$

$f(x)$ è un infinitesimo del 2° ordine per $x \to 0$

$f(x) \sim -5x^2$ parte principale

$f(x) = \sin x$ per $x \to 0$

$$\lim_{x \to 0} \frac{f(x)}{x^\alpha} = \lim_{x \to 0} \frac{\sin x}{x^\alpha} = 1 \quad \text{se } \alpha = 1$$

$f(x)$ è un infinitesimo del 1° ordine per $x \to 0$

$f(x) \sim x$ parte principale

$f(x) = (1 - \cos x)^2$ per $x \to 0$

$$\lim_{x \to 0} \frac{f(x)}{x^\alpha} = \lim_{x \to 0} \frac{x^4}{x^\alpha} \left( \frac{1 - \cos x}{x^2} \right)^2 = \frac{1}{4} \quad \text{se } \alpha = 4$$

$f(x)$ è infinitesimo del 4° ordine per $x \to 0$

$f(x) = \frac{1}{4} x^4$ parte principale

$$f(x) = \frac{1}{x^2+3x+1} \quad \text{per } x \to \infty$$

$$\lim_{x\to\infty} \frac{f(x)}{\left(\frac{1}{x}\right)^\alpha} = \lim_{x\to\infty} \frac{x^\alpha}{x^2\left(1+\frac{3}{x}+\frac{1}{x^2}\right)} = 1 \qquad \text{se } \alpha = 2$$

$f(x)$ è un infinitesimo del 2° ordine per $x \to \infty$

$f(x) \sim \frac{1}{x^2}$ parte principale

$$f(x) = \frac{x+1}{x^2+3} \quad \text{per } x \to \infty$$

$$\lim_{x\to\infty} \frac{f(x)}{\left(\frac{1}{x}\right)^\alpha} = \lim_{x\to\infty} \frac{x^{\alpha+1}\left(1+\frac{1}{x}\right)}{x^2\left(1+\frac{3}{x^2}\right)} = 1 \qquad \text{se } \alpha = 1$$

$f(x)$ è un infinitesimo del 1° ordine per $x \to \infty$

$f(x) \sim \frac{1}{x}$ parte principale

$f(x) = x^3 - x^2 + 1$ per $x \to \infty$

$$\lim_{x\to\infty} \frac{f(x)}{x^\alpha} = \lim_{x\to\infty} \frac{x^3 - x^2 + 1}{x^\alpha} = \lim_{x\to\infty} \frac{x^3}{x^\alpha}\left(1 - \frac{1}{x} + \frac{1}{x^3}\right) = 1 \qquad \text{se } \alpha = 3$$

$f(x)$ è un infinito del 3° ordine per $x \to \infty$

$x^3 \to$ parte principale

$f(x) = \dfrac{3-x}{x^4+x^2}$ per $x \to 0$

$$\lim_{x\to 0} \frac{f(x)}{\frac{1}{x^\alpha}} = \lim_{x\to 0} \frac{(3-x)\,x^\alpha}{(x^2+1)\,x^2} = 3 \qquad \text{se } \alpha = 2$$

$f(x)$ è un infinito del 2° ordine per $x \to 0$

$\dfrac{3}{x^2} \to$ parte principale

$$f(x) = \frac{2+3x}{\sqrt[3]{x}} \qquad \text{per} \quad x \to 0^+$$

$$\lim_{x\to 0^+} \frac{f(x)}{\frac{1}{x^\alpha}} = \lim_{x\to 0^+} \frac{(2+3x)\,x^\alpha}{x^{\frac{1}{3}}} = 2 \qquad \text{se} \quad \alpha = \frac{1}{3}$$

$f(x)$ è un infinito di ordine $\frac{1}{3}$ per $x \to 0^+$

$\frac{2}{\sqrt[3]{x}} \to$ parte principale

# Funzioni continue

$$f(x) = \begin{cases} 5x-1 & \text{per } x \geq 0 \\ 2x-a & \text{per } x < 0 \end{cases}$$

$f$ è continua in $\mathbb{R}$ se $\forall x_0 \in \mathbb{R}$ $\lim_{x \to x_0} f(x) = f(x_0)$

La funzione $y = 5x-1$ è continua per $x > 0$

La funzione $y = 2x-a$ è continua per $x < 0$

È necessario quindi studiare la continuità della funzione per $x \to 0$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (2x-a) = -a$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (5x-1) = -1$$

$$f(0) = 5 \cdot 0 - 1 = -1$$

Affinché $f$ sia continua per $x \to 0$, deve essere $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$.

$$-a = -1 \longrightarrow \boxed{a = 1}$$

$$f(x) = \begin{cases} 2x^2+\sqrt{x}+1 & \text{per } x \geq 0 \\ ax-3 & \text{per } x<0 \end{cases}$$

$f(0) = 1$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (ax-3) = -3$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (2x^2+\sqrt{x}+1) = 1$

Poiché per $x \to 0$ il limite destro e il limite sinistro non sono uguali, la funzione non è continua in $x=0$ qualunque sia il valore di $a$.

$$f(x) = \begin{cases} \dfrac{e^x-1}{\sin 3x} & \text{per } x>0 \\ a(x+1) & \text{per } x \leq 0 \end{cases}$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\cancel{x}\cdot\frac{e^x-1}{x}}{3\cancel{x}\cdot\frac{\sin 3x}{3x}} = \frac{1}{3}$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \left[a(x+1)\right] = a$$

$f(0) = a$

$f$ è continua in $x=0$ se $\boxed{a = \frac{1}{3}}$.

$f(x) = x^3 + x^2 - 4$ $\qquad$ $D = [1;2]$

$f$ è continua in $D$

$D$ è un intervallo chiuso e limitato

$f(1) = 1 + 1 - 4 = -2$

$f(2) = 8 + 4 - 4 = 8$

$f(1) \cdot f(2) < 0$

Sono soddisfatte le ipotesi del teorema di esistenza degli zeri, pertanto

$\exists\, c \in (1;2) \mid f(c) = 0$

Applicando il metodo di bisezione si trova che $c = 1{,}3145962$

$f(x) = \log(x-5) + x - 7 \qquad D = [6;7]$

$f$ è continua in $D$

$D$ è un intervallo chiuso e limitato

$f(6) = \log 1 + 6 - 7 = -1$

$f(7) = \log 2 > 0 \qquad (\log 2 \approx 0{,}693)$

$f(6) \cdot f(7) < 0$

Sono soddisfatte le ipotesi del teorema di esistenza degli zeri, pertanto

$\exists\, c \in (6;7) \mid f(c) = 0$

Applicando il metodo di bisezione si trova che

$c = 6{,}5571456$

$y = e^x \qquad D = [0; +\infty)$

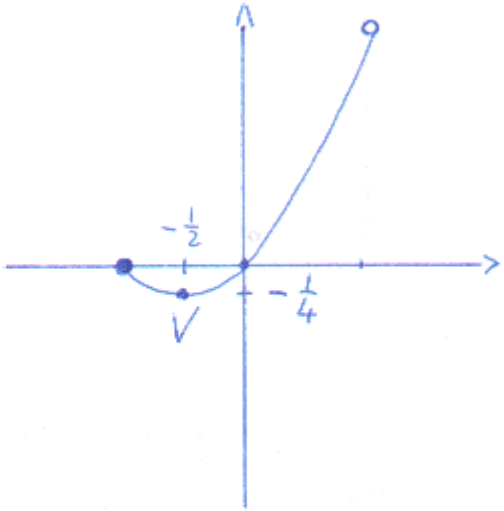
La funzione è continua e crescente in D

La funzione ammette minimo $y = 1$ per $x = 0$

Il teorema di Weierstrass non è applicabile perché D non è limitato.

$y = x^2 + x \qquad D = [-1; 1)$

D non è un intervallo chiuso, pertanto il teorema di Weierstrass non è applicabile.




Dal grafico della funzione (arco di parabola) si evince che la funzione ammette il valore minimo $y = -\frac{1}{4}$ in $x = -\frac{1}{2}$.
La funzione non ammette invece il valore massimo $(\sup(f(x)) = 2)$.

$y = \log x \qquad D = [1; e]$

$f$ è continua in $D$

$D$ è un intervallo chiuso e limitato

Sono soddisfatte le ipotesi del teorema di Weierstrass, che risulta pertanto applicabile.

Considerando che la funzione è crescente, risulta:

$y_{min} = \log 1 = 0 \qquad$ in $x = 1$;

$y_{max} = \log e = 1 \qquad$ in $x = e$.

# Punti di discontinuità

$$y = \frac{x^2+1}{x^2-9} \qquad x^2-9 \neq 0 \rightarrow x = \pm 3 \rightarrow D = \mathbb{R} - \{-3; +3\}$$

$$\lim_{x \to -3^-} \frac{x^2+1}{(x+3)(x-3)} = \frac{10}{0^+} = +\infty$$

$$\lim_{x \to -3^+} \frac{x^2+1}{(x+3)(x-3)} = \frac{10}{0^-} = -\infty$$

$x = -3$ punto di discontinuità di 2ª specie

$$\lim_{x \to 3^-} \frac{x^2+1}{x^2-9} = \frac{10}{0^-} = -\infty$$

$$\lim_{x \to 3^+} \frac{x^2+1}{x^2-9} = \frac{10}{0^+} = +\infty$$

$x = 3$ punto di discontinuità di 2ª specie

$$y = \frac{x^2-x-2}{x^2-3x+2} \qquad x^2-3x+2 \neq 0 \qquad (x-2)(x-1) \neq 0 \rightarrow x \neq 1,\ x \neq 2 \rightarrow D = \mathbb{R} - \{1;2\}$$

$$\lim_{x \to 1^-} \frac{x^2-x-2}{x^2-3x+2} = \lim_{x \to 1^-} \frac{\cancel{(x-2)}(x+1)}{\cancel{(x-2)}(x-1)} = \frac{2}{0^-} = -\infty$$

$$\lim_{x \to 1^+} \frac{x^2-x-2}{x^2-3x+2} = \lim_{x \to 1^+} \frac{\cancel{(x-2)}(x+1)}{\cancel{(x-2)}(x-1)} = \frac{2}{0^+} = +\infty$$

$x = 1$ (2ª specie)

$$\lim_{x \to 2^-} \frac{x^2-x-2}{x^2-3x+2} = \lim_{x \to 2^-} \frac{\cancel{(x-2)}(x+1)}{\cancel{(x-2)}(x-1)} = 3$$

$$\lim_{x \to 2^+} \frac{x^2-x-2}{x^2-3x+2} = \lim_{x \to 2^+} \frac{\cancel{(x-2)}(x+1)}{\cancel{(x-2)}(x-1)} = 3$$

$x = 2$ (3ª specie)

$$y = \frac{x}{e^x - 1} \qquad e^x - 1 \neq 0 \rightarrow x \neq 0 \rightarrow D = \mathbb{R} - \{0\}$$

$$\lim_{x \to 0^-} \frac{x}{e^x - 1} = \lim_{x \to 0^+} \frac{x}{e^x - 1} = 1 \qquad x = 0 \qquad (3^a \text{ specie})$$

$$y = e^{\frac{x}{x^2-1}} \qquad x^2 - 1 \neq 0 \rightarrow x \neq -1,\ x \neq 1 \rightarrow D = \mathbb{R} - \{-1; 1\}$$

$$\lim_{x \to -1^-} e^{\frac{x}{x^2-1}} = e^{\frac{-1}{0^+}} = e^{-\infty} = 0$$

$$\lim_{x \to -1^+} e^{\frac{x}{x^2-1}} = e^{\frac{-1}{0^-}} = e^{+\infty} = +\infty \qquad x = -1 \qquad (2^a \text{ specie})$$

$$\lim_{x \to 1^-} e^{\frac{x}{x^2-1}} = e^{\frac{1}{0^-}} = e^{-\infty} = 0$$

$$\lim_{x \to 1^+} e^{\frac{x}{x^2-1}} = e^{\frac{1}{0^+}} = e^{+\infty} = +\infty \qquad x = 1 \qquad (2^a \text{ specie})$$

$$y = x + \frac{|x|}{x} \qquad x \neq 0 \rightarrow D = \mathbb{R} - \{0\}$$

$$\lim_{x \to 0^-} \left( x + \frac{|x|}{x} \right) = \lim_{x \to 0^-} \left( x - \frac{x}{x} \right) = \lim_{x \to 0^-} (x-1) = -1$$

$$\lim_{x \to 0^+} \left( x + \frac{|x|}{x} \right) = \lim_{x \to 0^+} \left( x + \frac{x}{x} \right) = \lim_{x \to 0^+} (x+1) = +1$$

$x = 0$ (1ª specie)

$$y = \frac{\log(1+x)}{x} \qquad \begin{cases} 1+x > 0 \\ x \neq 0 \end{cases} \rightarrow \begin{cases} x > -1 \\ x \neq 0 \end{cases} \rightarrow D = (-1;0) \cup (0;+\infty)$$

$$\lim_{x \to 0^-} \frac{\log(1+x)}{x} = \lim_{x \to 0^+} \frac{\log(1+x)}{x} = 1$$

$x = 0$ (3ª specie)

$$\lim_{x \to -1^+} \frac{\log(1+x)}{x} = +\infty$$

$x = -1$ (2ª specie)

$$y = \frac{|x+1|}{x+1} e^{\frac{1}{x-2}} \qquad x \neq -1 \,,\; x \neq 2 \;\rightarrow\; D = \mathbb{R} - \{-1; 2\}$$

$$\lim_{x \to -1^-} \frac{|x+1|}{x+1} e^{\frac{1}{x-2}} = \lim_{x \to -1^-} \frac{-\cancel{(x+1)}}{\cancel{x+1}} e^{\frac{1}{x-2}} = -e^{-\frac{1}{3}}$$

$$\lim_{x \to -1^+} \frac{|x+1|}{x+1} e^{\frac{1}{x-2}} = \lim_{x \to -1^+} \frac{\cancel{x+1}}{\cancel{x+1}} e^{\frac{1}{x-2}} = e^{-\frac{1}{3}}$$

$x = -1$ (1ª specie)

$$\lim_{x \to 2^-} \frac{|x+1|}{x+1} e^{\frac{1}{x-2}} = \lim_{x \to 2^-} \frac{\cancel{x+1}}{\cancel{x+1}} e^{\frac{1}{x-2}} = e^{\frac{1}{0^-}} = e^{-\infty} = 0$$

$$\lim_{x \to 2^+} \frac{|x+1|}{x+1} e^{\frac{1}{x-2}} = \lim_{x \to 2^+} \frac{\cancel{x+1}}{\cancel{x+1}} e^{\frac{1}{x-2}} = e^{\frac{1}{0^+}} = e^{+\infty} = +\infty$$

$x = 2$ (2ª specie)

$$y = x \sin \frac{1}{x} \qquad x \neq 0 \;\rightarrow\; D = \mathbb{R} - \{0\}$$

$$\lim_{x \to 0^-} x \sin \frac{1}{x} = \lim_{t \to -\infty} \frac{\sin t}{t} = 0$$

$$t = \frac{1}{x}$$

$$\lim_{x \to 0^+} x \sin \frac{1}{x} = \lim_{t \to +\infty} \frac{\sin t}{t} = 0$$

$x = 0$ (3ª specie)

$$y = \begin{cases} x+2 & \text{per } x>0 \\ 3 & \text{per } x=0 \\ \dfrac{\sin x}{x} & \text{per } x<0 \end{cases}$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x+2) = 2$$

$x=0$ (1ª specie)

$$y = \begin{cases} \dfrac{e^x-1}{x} & \text{per } x<0 \\ 1+x & \text{per } x>0 \end{cases}$$

$f(0)\ \nexists$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{e^x-1}{x} = 1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (1+x) = 1$$

$x=0$ (3ª specie)

$$y = \frac{1-\cos x}{\cos x - \cos 2x}$$

$$\cos x - \cos 2x \neq 0$$

$$\cos x - 2\cos^2 x + 1 \neq 0$$

$$2\cos^2 x - \cos x - 1 \neq 0$$

$$(\cos x - 1)(2\cos x + 1) \neq 0$$

$$\cos x \neq 1 \quad \rightarrow \quad x \neq 2K\pi$$

$$\cos x \neq -\frac{1}{2} \quad \rightarrow \quad x \neq \pm\frac{2}{3}\pi + 2K\pi$$

$$\lim_{x\to 0} \frac{1-\cos x}{\cos x - \cos 2x} = \lim_{x\to 0} \frac{\cancel{1-\cos x}}{-\cancel{(1-\cos x)}(2\cos x+1)} = -\frac{1}{3}$$

$x = 2K\pi$ (3ª specie)

$$\lim_{x\to \frac{2}{3}\pi^-} \frac{1-\cos x}{\cos x - \cos 2x} = \lim_{x\to \frac{2}{3}\pi^-} \frac{-1}{2\cos x + 1} = \frac{-1}{0^+} = -\infty$$

$$\lim_{x\to \frac{2}{3}\pi^+} \frac{1-\cos x}{\cos x - \cos 2x} = \lim_{x\to \frac{2}{3}\pi^+} \frac{-1}{2\cos x + 1} = \frac{-1}{0^-} = +\infty$$

$x = \pm\frac{2}{3}\pi + 2K\pi$ (2ª specie)

$$f(x) = \begin{cases} ax+4 & \text{per } x>2 \\ 4+x & \text{per } x<2 \end{cases}$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (4+x) = 6$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (ax+4) = 2a+4$$

Si ha una discontinuità eliminabile se

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$$

$$6 = 2a+4$$

$$-2a = -2 \quad \rightarrow \quad a = 1$$

# Derivate e loro applicazione

# Definizione di derivata

$$y = x^2 + 1 \qquad x = 0 \qquad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(0) = \lim_{h \to 0} \frac{h^2 + 1 - 1}{h} = \lim_{h \to 0} \frac{h^2}{h} = \lim_{h \to 0} h = 0$$

$$y = \frac{2x-3}{x} \qquad x = 1$$

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\frac{2+2h-3}{1+h} + 1}{h} = \lim_{h \to 0} \frac{2h \cancel{-1} \cancel{+1} + h}{h} = \lim_{h \to 0} \frac{3\cancel{h}}{\cancel{h}} = 3$$

$$y = \frac{x}{x-1} \qquad x = -1$$

$$f'(-1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h \to 0} \frac{\frac{-1+h}{-2+h} - \frac{1}{2}}{h} = \lim_{h \to 0} \frac{2h \cancel{-2} - h \cancel{+2}}{2h(h-2)} = \lim_{h \to 0} \frac{\cancel{h}}{2\cancel{h}(h-2)} = -\frac{1}{4}$$

$y = \log x \qquad x = 1$

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\log(1+h)}{h} = \log e = 1$$

$y = e^x \qquad x = 0$

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{e^h - 1}{h} = \log e = 1$$

$y = \sin x \qquad x = \frac{\pi}{2}$

$$f'\left(\frac{\pi}{2}\right) = \lim_{h \to 0} \frac{f\left(\frac{\pi}{2}+h\right) - f\left(\frac{\pi}{2}\right)}{h} = \lim_{h \to 0} \frac{\sin\left(\frac{\pi}{2}+h\right) - 1}{h} =$$

$$= \lim_{h \to 0} \frac{\cos h - 1}{h} = \lim_{h \to 0} \left[(-h) \cdot \frac{1 - \cos h}{h^2}\right] = 0$$

$y = \cos x \qquad x = \pi$

$$f'(\pi) = \lim_{h\to 0} \frac{f(\pi+h) - f(\pi)}{h} = \lim_{h\to 0} \frac{\cos(\pi+h) + 1}{h} =$$

$$= \lim_{h\to 0} \frac{-\cos h + 1}{h} = \lim_{h\to 0} \left[ h \cdot \frac{1-\cos h}{h^2} \right] = 0$$

$y = x^2 + x \qquad D = \mathbb{R}$

$$y' = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\cancel{x^2} + 2hx + h^2 + \cancel{x} + h - \cancel{x^2} - \cancel{x}}{h} = \lim_{h\to 0} \frac{\cancel{h}(2x+1+h)}{\cancel{h}} =$$

$$= \lim_{h\to 0} (2x+1+h) = 2x+1 \qquad D' = \mathbb{R}$$

$y = \dfrac{x+1}{x-2} \qquad D = \mathbb{R} - \{2\}$

$$y' = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{x+1+h}{x-2+h} - \frac{x+1}{x-2}}{h} = \lim_{h\to 0} \frac{\cancel{x^2} + \cancel{x} + \cancel{hx} - \cancel{2x} - \cancel{2} - 2h - \cancel{x^2} - \cancel{x} + \cancel{2x} + \cancel{2} - \cancel{hx} - h}{h(x-2)(x-2+h)} =$$

$$= \lim_{h\to 0} \frac{-3\cancel{h}}{\cancel{h}(x-2)(x-2+h)} = -\frac{3}{(x-2)^2} \qquad D' = \mathbb{R} - \{2\}$$

$y = e^{3x}$ $\qquad D = \mathbb{R}$

$$y' = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{e^{3(x+h)} - e^{3x}}{h} = \lim_{h\to 0} \frac{e^{3x} \cdot e^{3h} - e^{3x}}{h} =$$

$$= \lim_{h\to 0} \left[ 3e^{3x} \frac{e^{3h} - 1}{3h} \right] = 3e^{3x}$$

$D' = \mathbb{R}$

$y = \sin 2x$ $\qquad D = \mathbb{R}$

$$y' = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(2x+2h) - \sin 2x}{h} =$$

$$= \lim_{h\to 0} \frac{2 \cos \frac{2x+2h+2x}{2} \sin \frac{2x+2h-2x}{2}}{h} =$$

$$= \lim_{h\to 0} \frac{2 \cos(2x+h) \sin h}{h} = 2 \cos 2x$$

$D' = \mathbb{R}$

# Calcolo di derivate

$$y = (\sin x)^{\cos x}$$

$$\log y = \cos x \log \sin x$$

$$\frac{y'}{y} = -\sin x \log \sin x + \cos x \cdot \frac{1}{\sin x} \cdot \cos x$$

$$y' = (\sin x)^{\cos x - 1} \left[ \cos^2 x - \sin^2 x \log \sin x \right]$$

$$y = (\sin x)^{x}$$

$$\log y = x \log \sin x$$

$$\frac{y'}{y} = \log \sin x + x \frac{\cos x}{\sin x}$$

$$y' = (\sin x)^{x} \left( \log \sin x + x \operatorname{cotg} x \right)$$

Oppure applicare direttamente la seguente formula:

$$\mathbf{D}\left[f(x)\right]^{g(x)} = \left[f(x)\right]^{g(x)} \cdot \left[ g'(x) \cdot \ln\left[f(x)\right] + \frac{g(x) \cdot f'(x)}{f(x)} \right]$$

$$y = x^{2x}$$

$$\log y = \log x^{2x}$$

$$\log y = 2x \log x$$

$$\frac{y'}{y} = 2\log x + 2x \cdot \frac{1}{x}$$

$$\frac{y'}{y} = 2(\log x + 1)$$

$$y' = 2x^{2x}(\log x + 1)$$

$$y = x^{2x^2+1}$$

$$\log y = \log x^{2x^2+1}$$

$$\log y = (2x^2+1)\log x$$

$$\frac{y'}{y} = 4x\log x + \frac{2x^2+1}{x}$$

$$y' = x^{2x^2} \cdot x\left(4x\log x + \frac{2x^2+1}{x}\right)$$

$$y' = x^{2x^2}\left(4x^2\log x + 2x^2 + 1\right)$$

Oppure applicare direttamente la seguente formula:

$$\mathbf{D}\left[f(x)\right]^{g(x)} = \left[f(x)\right]^{g(x)} \cdot \left[g'(x) \cdot \ln\left[f(x)\right] + \frac{g(x) \cdot f'(x)}{f(x)}\right]$$

$$y = (x^3+1)^{x^2-1}$$

$$\log y = (x^2-1)\log(x^3+1)$$

$$\frac{y'}{y} = 2x\log(x^3+1) + \frac{3x^2(x^2-1)}{x^3+1}$$

$$y' = (x^3+1)^{x^2-1}\left[2x\log(x^3+1) + \frac{3x^2\cancel{(x+1)}(x-1)}{\cancel{(x+1)}(x^2-x+1)}\right]$$

$$y' = (x^3+1)^{x^2-1}\left[2x\log(x^3+1) + \frac{3x^2(x-1)}{x^2-x+1}\right]$$

$$y = (\log x)^{\log x}$$

$$\log y = \log x \cdot \log\log x$$

$$\frac{y'}{y} = \frac{1}{x}\log\log x + \frac{1}{x}\cdot\frac{1}{\cancel{\log x}}\cdot\cancel{\log x}$$

$$y' = \frac{1}{x}(\log x)^{\log x}\left[\log\log x + 1\right]$$

Oppure applicare direttamente la seguente formula:

$$\boxed{\mathbf{D}\,[f(x)]^{g(x)} = [f(x)]^{g(x)}\cdot\left[g'(x)\cdot\ln[f(x)] + \frac{g(x)\cdot f'(x)}{f(x)}\right]}$$

$$y = x^{\frac{1}{x}}$$

$$\log y = \log x^{\frac{1}{x}}$$

$$\log y = \frac{\log x}{x}$$

$$\frac{y'}{y} = \frac{1 - \log x}{x^2}$$

$$y' = x^{\frac{1}{x}} \frac{1 - \log x}{x^2}$$

$$y = \sin x^{x^2+1}$$

$$y = \sin z$$

$$z = x^{x^2+1}$$

$$\log z = \log (x)^{x^2+1}$$

$$\log z = (x^2+1) \log x$$

$$\frac{z'}{z} = 2x \log x + \frac{x^2+1}{x}$$

$$z' = x^{x^2+1} \left(2x \log x + \frac{x^2+1}{x}\right)$$

$$y' = \cos x^{x^2+1} \cdot x^{x^2+1} \left(2x \log x + \frac{x^2+1}{x}\right)$$

$$y' = x^{x^2} \cos x^{x^2+1} \left(2x^2 \log x + x^2+1\right)$$

$$y = e^{2x} \log(1+x)$$

$$y' = 2e^{2x} \log(1+x) + e^{2x} \frac{1}{1+x} = e^{2x}\left[2\log(1+x) + \frac{1}{1+x}\right]$$

$$y = \arcsin \frac{x-1}{x+1}$$

$$y' = \frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}} \cdot \frac{x+1-x+1}{(x+1)^2} = \frac{2}{x+1} \cdot \frac{1}{\sqrt{(x+1)^2-(x-1)^2}} =$$

$$= \frac{2}{x+1} \frac{1}{\sqrt{\cancel{x^2}+2x+\cancel{1}-\cancel{x^2}+2x-\cancel{1}}} = \frac{\cancel{2}^{1}}{\cancel{2}_{1}(x+1)\sqrt{x}} = \frac{1}{(x+1)\sqrt{x}}$$

$$y = \frac{2x+3}{x^2-x} \qquad y' = \frac{2(x^2-x)-(2x-1)(2x+3)}{(x^2-x)^2} = \frac{2x^2-2x-4x^2-6x+2x+3}{(x^2-x)^2} = \frac{-2x^2-6x+3}{(x^2-x)^2}$$

$$y = \frac{x^4}{x^3+1} \qquad y' = \frac{4x^3(x^3+1)-3x^2 \cdot x^4}{(x^3+1)^2} = \frac{4x^6+4x^3-3x^6}{(x^3+1)^2} = \frac{x^6+4x^3}{(x^3+1)^2}$$

$$y = \frac{3}{\sqrt[3]{2+x^2}} \qquad y' = -3 \cdot \frac{1}{3}\left(2+x^2\right)^{-\frac{4}{3}} \cdot 2x = -\frac{2x}{\sqrt[3]{(2+x^2)^4}}$$

$$y = \sqrt{\frac{x+2}{3x}} \qquad y' = \frac{1}{2\sqrt{\frac{x+2}{3x}}} \; \frac{3x-3x-6}{9x^2} = -\frac{1}{3x^2}\sqrt{\frac{3x}{x+2}}$$

$$y = \frac{\cos x}{1+\cos^2 x} \qquad y' = \frac{-\sin x\left(1+\cos^2 x\right) + 2\cos x \cdot \sin x \cdot \cos x}{\left(1+\cos^2 x\right)^2} = \frac{\cos^2 x \sin x - \sin x}{\left(1+\cos^2 x\right)^2}$$

$$y' = \frac{-\sin x\left(1-\cos^2 x\right)}{\left(1+\cos^2 x\right)^2} = \frac{-\sin^3 x}{\left(1+\cos^2 x\right)^2}$$

$$y = \log \sin\sqrt{2x+1} \qquad y' = \frac{1}{\sin\sqrt{2x+1}} \cdot \cos\sqrt{2x+1} \cdot \frac{2}{2\sqrt{2x+1}} = \frac{\operatorname{cotg}\sqrt{2x+1}}{\sqrt{2x+1}}$$

$$y = \operatorname{arctg} x + \log\left(1+x^2\right) \qquad y' = \frac{1}{1+x^2} + \frac{2x}{1+x^2} = \frac{1+2x}{1+x^2}$$

$$y = \log\left(x^2-3x+1\right)^2 \qquad y' = \frac{2\left(x^2-3x+1\right)\left(2x-3\right)}{\left(x^2-3x+1\right)^2} = \frac{2\left(2x-3\right)}{x^2-3x+1}$$

$$y = \frac{2-3\sin x}{\sin x-\cos x} \qquad y' = \frac{-3\cos x\left(\sin x-\cos x\right)-\left(\cos x+\sin x\right)\left(2-3\sin x\right)}{\left(\sin x-\cos x\right)^2}$$

$$y' = \frac{-3\sin x\cos x+3\cos^2 x-2\cos x+3\sin x\cos x-2\sin x+3\sin^2 x}{\left(\sin x-\cos x\right)^2}$$

$$y' = \frac{3-2\sin x-2\cos x}{\left(\sin x-\cos x\right)^2}$$

$$y = \log\frac{1-x}{1+x} - 2\log\left(1-x^2\right) \qquad y' = \frac{1}{\frac{1-x}{1+x}}\cdot\frac{-1-x-1+x}{\left(1+x\right)^2} - 2\cdot\frac{\left(-2x\right)}{1-x^2} = \frac{2\left(2x-1\right)}{1-x^2}$$

$$y = \sin 2\sqrt{x} + 2\sqrt{\sin x} \qquad y' = \cos 2\sqrt{x}\cdot 2\frac{1}{2\sqrt{x}} + 2\frac{\cos x}{2\sqrt{\sin x}} = \frac{\cos 2\sqrt{x}}{\sqrt{x}} + \frac{\cos x}{\sqrt{\sin x}}$$

$$y = x\,\mathrm{arctg}\,\frac{1}{x} \qquad y' = \mathrm{arctg}\,\frac{1}{x} + x\,\frac{1}{1+\frac{1}{x^2}}\cdot\left(-\frac{1}{x^2}\right) = \mathrm{arctg}\,\frac{1}{x} - \frac{x}{x^2+1}$$

$$y = \operatorname{arctg} \frac{2x}{1-x^2}$$

$$y' = \frac{1}{1+\frac{4x^2}{(1-x^2)^2}} \cdot \frac{2(1-x^2)+2x\cdot 2x}{(1-x^2)^2} = \frac{2-2x^2+4x^2}{(1-x^2)^2+4x^2}$$

$$y' = \frac{2+2x^2}{1-2x^2+x^4+4x^2} = \frac{2(1+x^2)}{1+2x^2+x^4} = \frac{2(1+x^2)}{(1+x^2)^2}$$

$$y' = \frac{2}{1+x^2}$$

$$y = e^x\left[x^2-2(x-1)\right] = e^x(x^2-2x+2)$$

$$y' = e^x(x^2-2x+2) + e^x(2x-2) = e^x(x^2-2x+2+2x-2) = x^2e^x$$

$$y = \arcsin\sqrt{1-x^2}$$

$$y' = \frac{1}{\sqrt{1-1+x^2}} \cdot \frac{-x}{\sqrt{1-x^2}} = \frac{-x}{|x|\sqrt{1-x^2}}$$

$$y = \operatorname{tg} e^{x^2+3}$$

$$y' = \frac{2x\,e^{x^2+3}}{\cos^2 e^{x^2+3}}$$

$$y = (x+2)^{\log x}$$

$$\log y = \log (x+2)^{\log x}$$

$$\log y = \log x \log (x+2)$$

$$\frac{y'}{y} = \frac{1}{x} \log (x+2) + \frac{1}{x+2} \log x$$

$$y' = (x+2)^{\log x} \left\{ \frac{\log (x+2)}{x} + \frac{\log x}{x+2} \right\}$$

$$y = \left(\cos 2x - \cos^2 x\right)^3 = \left(\cos^2 x - \sin^2 x - \cos^2 x\right)^3 = -\sin^6 x$$

$$y' = -6 \sin^5 x \cos x$$

$$y = x + \sin 2x \cos 2x = x + \frac{1}{2} \sin 4x$$

$$y' = 1 + \frac{1}{2} \cdot 4 \cos 4x = 1 + 2 \cos 4x$$

$$y = e^{-x}\left(x^3 + \frac{x^4}{4}\right)$$

$$y' = -e^{-x}\left(x^3 + \frac{x^4}{4}\right) + e^{-x}\left(3x^2 + x^3\right)$$

$$y' = e^{-x}\left(-x^3 - \frac{x^4}{4} + 3x^2 + x^3\right) = e^{-x}\left(3x^2 - \frac{x^4}{4}\right)$$

$$y' = \frac{e^{-x}\, x^2\,(12 - x^2)}{4}$$

$$y = \frac{\sin 2x}{2\sin x\,(1+\cos^2 x)} = \frac{\cos x}{1+\cos^2 x}$$

$$y' = \frac{-\sin x\,(1+\cos^2 x) + 2\cos x \sin x \cdot \cos x}{(1+\cos^2 x)^2}$$

$$y' = \frac{-\sin x - \sin x \cos^2 x + 2\sin x \cos^2 x}{(1+\cos^2 x)^2} = \frac{\sin x \cos^2 x - \sin x}{(1+\cos^2 x)^2}$$

$$y' = \frac{-\sin x\,(1-\cos^2 x)}{(1+\cos^2 x)^2} = \frac{-\sin^3 x}{(1+\cos^2 x)^2}$$

$$y = \log|x| + \frac{1}{3x^3}$$

$$y' = \frac{1}{x} - \frac{1}{x^4} = \frac{x^3 - 1}{x^4}$$

$$y = \sqrt{\frac{1+\sin x}{1-\sin x}}$$

$$y' = \frac{1}{2}\,\frac{1}{\sqrt{\frac{1+\sin x}{1-\sin x}}}\cdot\frac{\cos x - \cancel{\sin x\cos x} + \cos x + \cancel{\sin x\cos x}}{(1-\sin x)^2}$$

$$y' = \frac{1}{2}\,\frac{2\cos x}{\sqrt{1-\sin^2 x}\,(1-\sin x)} = \frac{\cos x}{|\cos x|\cdot(1-\sin x)}$$

$$y = 2\,\mathrm{arctg}\,x + \frac{4x+1}{2x^2}$$

$$y' = \frac{2}{1+x^2} + \frac{8x^2-16x^2-4x}{4x^4} = \frac{2}{1+x^2} - \frac{\cancel{4x}^{1}(2x+1)}{\cancel{4}_{1}x^{\cancel{4}\,3}}$$

$$y' = \frac{2}{1+x^2} - \frac{2x+1}{x^3} = \frac{\cancel{2x^3}-2x-\cancel{2x^3}-1-x^2}{x^3(1+x^2)} = -\frac{(x+1)^2}{x^3(1+x^2)}$$

$$y = \log\left(2x+\sqrt{1+4x^2}\right)$$

$$y' = \frac{1}{2x+\sqrt{1+4x^2}}\left(2+\frac{\cancel{8}^{4}x}{\cancel{2}_{1}\sqrt{1+4x^2}}\right) = \frac{2\,\cancel{\left(\sqrt{1+4x^2}+2x\right)}}{\cancel{\left(2x+\sqrt{1+4x^2}\right)}\,\sqrt{1+4x^2}}$$

$$y' = \frac{2}{\sqrt{1+4x^2}}$$

$$y = \frac{x^2 \log x - x \log x + 1}{x \log x + 1}$$

$$y' = \frac{\left(2x \log x + x^2 \cdot \frac{1}{x} - \log x - 1\right)(x \log x + 1) - (\log x + 1)(x^2 \log x - x \log x + 1)}{(x \log x + 1)^2}$$

$$y' = \frac{2x^2 \log^2 x + 2x \log x + \cancel{x^2 \log x} + x - \cancel{x \log^2 x} - \log x - \cancel{x \log x} - 1 - x^2 \log^2 x + \cancel{x \log^2 x} - \log x - \cancel{x^2 \log x} + \cancel{x \log x} - 1}{(x \log x + 1)^2}$$

$$y' = \frac{x^2 \log^2 x + 2x \log x + x - 2 \log x - 2}{(x \log x + 1)^2}$$

$$y = x \operatorname{arctg} x - \log \sqrt{1+x^2}$$

$$y' = \operatorname{arctg} x + \frac{x}{1+x^2} - \frac{1}{\sqrt{1+x^2}} \cdot \frac{\cancel{2}x}{\cancel{2}\sqrt{1+x^2}}$$

$$y' = \operatorname{arctg} x + \frac{x}{1+x^2} - \frac{x}{1+x^2} = \operatorname{arctg} x$$

$$y = e^x \log \cos\sqrt{x}$$

$$y' = e^x \log\cos\sqrt{x} + e^x \frac{(-\sin\sqrt{x})\cdot\frac{1}{2\sqrt{x}}}{\cos\sqrt{x}} = e^x\left\{\log\cos\sqrt{x} - \frac{1}{2\sqrt{x}}\,\mathrm{tg}\sqrt{x}\right\}$$

$$y = \mathrm{arctg}\left(1 - \frac{4x}{x^2+2x-1}\right) = \mathrm{arctg}\,\frac{x^2-2x-1}{x^2+2x-1}$$

$$y' = \frac{1}{1+\frac{(x^2-2x-1)^2}{(x^2+2x-1)^2}}\;\frac{(2x-2)(x^2+2x-1)-(2x+2)(x^2-2x-1)}{(x^2+2x-1)^2}$$

$$y' = \frac{\cancel{2x^3}+4x^2-\cancel{2x}-2x^2-\cancel{4x}+2-\cancel{2x^3}+4x^2+\cancel{2x}-2x^2+\cancel{4x}+2}{x^4+4x^2+1-2x^2+\cancel{4x^3}-\cancel{4x}+x^4+4x^2+1-2x^2-\cancel{4x^3}+\cancel{4x}}$$

$$y' = \frac{4x^2+4}{2x^4+4x^2+2} = \frac{4(x^2+1)}{2(x^4+2x^2+1)} = \frac{2\cancel{(x^2+1)}}{(x^2+1)^{\cancel{2}}} = \frac{2}{x^2+1}$$

$$y = 2\,\mathrm{arctg}\,x + \frac{4x+1}{2x^2}$$

$$y' = \frac{2}{1+x^2} + \frac{8x^2-16x^2-4x}{4x^4} = \frac{2}{1+x^2} - \frac{\cancel{4x}(2x+1)}{\cancel{4}x^{\cancel{4}\,3}}$$

$$y' = \frac{2}{1+x^2} - \frac{2x+1}{x^3} = \frac{\cancel{2x^3}-2x-\cancel{2x^3}-1-x^2}{x^3(1+x^2)} = -\frac{(x+1)^2}{x^3(1+x^2)}$$

$$y = e^{-x}\left(x^3+\frac{x^4}{4}\right)$$

$$y' = -e^{-x}\left(x^3+\frac{x^4}{4}\right) + e^{-x}\left(3x^2+x^3\right)$$

$$y' = e^{-x}\left(-x^3-\frac{x^4}{4}+3x^2+x^3\right) = e^{-x}\left(3x^2-\frac{x^4}{4}\right)$$

$$y' = \frac{e^{-x}x^2(12-x^2)}{4}$$

$$y = \sqrt{1-e^{2\cos x}} + \arcsin\sqrt{1-e^{2\cos x}}$$

$$y' = \frac{\cancel{2}\sin x\, e^{2\cos x}}{\cancel{2}\sqrt{1-e^{2\cos x}}} + \frac{1}{\sqrt{1-1+e^{2\cos x}}}\cdot\frac{\cancel{2}\sin x\, e^{2\cos x}}{\cancel{2}\sqrt{1-e^{2\cos x}}}$$

$$y' = \frac{\sin x\, e^{2\cos x}}{\sqrt{1-e^{2\cos x}}} + \frac{\sin x\, e^{\cos x}}{\sqrt{1-e^{2\cos x}}} = \frac{\sin x\, e^{\cos x}\left(e^{\cos x}+1\right)}{\sqrt{1-e^{2\cos x}}}$$

# Calcolo derivate successive

$$y = 2x^4 - 3x^2 - 1$$
$$y' = 8x^3 - 6x$$
$$y'' = 24x^2 - 6$$

$$y = (x-1)^4$$
$$y' = 4(x-1)^3$$
$$y'' = 12(x-1)^2$$


$$y = 2x^4 - 3x^2 - 1$$
$$y' = 8x^3 - 6x$$
$$y'' = 24x^2 - 6$$

$$y = (x-1)^4$$
$$y' = 4(x-1)^3$$
$$y'' = 12(x-1)^2$$

$$y = \log x$$

$$y' = \frac{1}{x}$$

$$y'' = -\frac{1}{x^2}$$

$$y = \sqrt{x}$$

$$y' = \frac{1}{2\sqrt{x}}$$

$$y'' = -\frac{1}{4x\sqrt{x}}$$

$$y = \frac{1}{2}\log^4 x$$

$$y' = \frac{2}{x}\log^3 x$$

$$y'' = -\frac{2}{x^2}\log^3 x + \frac{6}{x^2}\log^2 x$$

$$y'' = \frac{2\log^2 x\,(3 - \log x)}{x^2}$$

$$y = \frac{x}{x-1}$$

$$y' = \frac{x-1-x}{(x-1)^2} = -\frac{1}{(x-1)^2}$$

$$y'' = \frac{2}{(x-1)^3}$$

$$y = \frac{3x}{2x-1}$$

$$y' = \frac{6x-3-6x}{(2x-1)^2} = -\frac{3}{(2x-1)^2}$$

$$y'' = \frac{12}{(2x-1)^3}$$

$$y = x^4 + e^x$$

$$y' = 4x^3 + e^x$$

$$y'' = 12x^2 + e^x$$

$$y = (2x+1)e^{-x}$$

$$y' = 2e^{-x} + (2x+1)(-e^{-x})$$

$$y' = e^{-x}(1-2x)$$

$$y'' = -e^{-x}(1-2x) - 2e^{-x}$$

$$y'' = e^{-x}(2x-3)$$

$$y = x^2 e^{2x}$$

$$y' = 2xe^{2x} + 2x^2 e^{2x}$$

$$y' = e^{2x}(2x+2x^2)$$

$$y'' = 2e^{2x}(2x+2x^2) + e^{2x}(2+4x)$$

$$y'' = e^{2x}(4x^2+8x+2)$$

$$y'' = 2e^{2x}(2x^2+4x+1)$$

$$y = 2\sin x - \cos x$$

$$y' = 2\cos x + \sin x$$

$$y'' = -2\sin x + \cos x$$

$$y = x^2 - \sin^2 x$$

$$y' = 2x - 2\sin x \cos x$$

$$y' = 2x - \sin 2x$$

$$y'' = 2 - 2\cos 2x = 2(1 - \cos 2x)$$

$$y = \operatorname{arctg} x$$

$$y' = \frac{1}{1+x^2}$$

$$y'' = \frac{-2x}{(1+x^2)^2}$$

$$y = x \sin x$$

$$y' = \sin x + x\cos x$$

$$y'' = \cos x + \cos x - x\sin x$$

$$y'' = 2\cos x - x\sin x$$

# Differenziale di una funzione

$$y = x^3 - 2x + 5$$

$$f'(x) = 3x^2 - 2$$

$$dy = f'(x)\,dx = (3x^2 - 2)\,dx$$

$$y = \frac{x-3}{x}$$

$$f'(x) = \frac{x - x + 3}{x^2} = \frac{3}{x^2}$$

$$dy = \frac{3}{x^2}\,dx$$

$$y = \sqrt{x - x^3}$$

$$f'(x) = \frac{1 - 3x^2}{2\sqrt{x - x^3}}$$

$$dy = \frac{1 - 3x^2}{2\sqrt{x - x^3}}\,dx$$

$$y = \log^2 x + \log x$$

$$f'(x) = \frac{1}{x}(2\log x + 1)$$

$$dy = \frac{2\log x + 1}{x}\,dx$$

$$y = e^{-x} + 2e^{x}$$

$$y' = -e^{-x} + 2e^{x}$$

$$dy = \left(2e^{x} - e^{-x}\right)dx$$

$$y = \sin^3 2x$$

$$y' = 6\sin^2 2x \cos 2x$$

$$dy = 6\sin^2 2x \cos 2x \, dx$$

$$y = x^2 e^{x}$$

$$y' = 2xe^{x} + x^2 e^{x} = xe^{x}(x+2)$$

$$dy = xe^{x}(x+2)dx$$

$$y = \sin x - \cos^2 x$$

$$y' = \cos x + 2\sin x \cos x = \cos x + \sin 2x$$

$$dy = (\cos x + \sin 2x)\,dx$$

$$y = \frac{x^3+1}{2-x^3} \qquad x = 1$$

$$f'(x) = \frac{6x^2 - 3x^5 + 3x^5 + 3x^2}{(2-x^3)^2} = \frac{9x^2}{(2-x^3)^2}$$

$$f'(1) = 9 \qquad df(1) = 9\,dx$$

$$y = \sin x + \cos x \qquad x = \frac{\pi}{2}$$

$$f'(x) = \cos x - \sin x$$

$$f'\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} - \sin\frac{\pi}{2} = -1 \qquad df\left(\frac{\pi}{2}\right) = f'\left(\frac{\pi}{2}\right)dx = -dx$$

$$y = \log x + e^{x-1} \qquad x = 1$$

$$f'(x) = \frac{1}{x} + e^{x-1} \qquad f'(1) = 2 \qquad df(1) = 2dx$$

$$y = e^{\sin x} \qquad x = \frac{\pi}{2}$$

$$f'(x) = \cos x\, e^{\sin x} \qquad f'\left(\frac{\pi}{2}\right) = 0 \qquad df\left(\frac{\pi}{2}\right) = 0$$

$$y = \frac{e^x}{e^x - 1} \qquad x = 1$$

$$f'(x) = \frac{e^{2x} - e^x - e^{2x}}{(e^x - 1)^2} = -\frac{e^x}{(e^x - 1)^2} \qquad f'(1) = -\frac{e}{(e-1)^2} \qquad df(1) = -\frac{e}{(e-1)^2}\,dx$$

$$y = x^2 \qquad x = 1 \qquad \Delta x = 10^{-3} \qquad \Delta y = ? \qquad dy = ?$$

$$\Delta y = f(x + \Delta x) - f(x) = f(1{,}001) - f(1) = (1{,}001)^2 - 1 = 2{,}001 \cdot 0{,}001 = 2{,}001 \cdot 10^{-3}$$

$$dy = f'(1)\,dx = 2\,\Delta x = 2 \cdot 10^{-3}$$

$$\Delta y - dy = 10^{-6}$$

$y = \log x \qquad x_1 = 1 \qquad x_2 = 1{,}001 \qquad \Delta y = ? \qquad dy = ?$

$$\Delta y = f(x_2) - f(x_1) = \log 1{,}001 - \log 1 = 9{,}995003\ldots \cdot 10^{-4}$$

$$\Delta x = x_2 - x_1 = 10^{-3} \qquad f'(x) = \frac{1}{x}$$

$$dy = f'(1)\,dx = \Delta x = 10^{-3}$$

$$\Delta y - dy = \log 1{,}001 - \frac{1}{1000} \approx -4{,}997 \cdot 10^{-7}$$

$y = x^2 + x + 1 \qquad x_1 = 1 \qquad x_2 = 1 + 10^{-3} \qquad \Delta y = ? \qquad dy = ?$

$$\Delta y = f(x_2) - f(x_1) = (1{,}001)^2 + 1{,}01 \cancel{+1} - 1 - 1 \cancel{-1} = 0{,}012001$$

$$f'(x) = 2x + 1 \qquad \Delta x = 10^{-3} \qquad dx = \Delta x$$

$$dy = f'(1)\,dx = 3 \cdot 10^{-3} \qquad \Delta y - dy = 0{,}009001 = 9{,}001 \cdot 10^{-3}$$

# Applicazione derivate: tangente a una curva

$y = 4x^3 \qquad (x=-1)$

$f(-1) = -4 \qquad P(-1;-4)$

$y' = 12x^2 \qquad m = f'(-1) = 12$

eq. retta tangente in P:

$$y - y_P = m(x - x_P)$$

$$y + 4 = 12(x+1)$$

$$y = 12x + 8$$

$y = 2x^2 + 2x - 3 \qquad (x=1)$

$f(1) = 1 \qquad P(1;1)$

$f'(x) = 4x + 2 \qquad m = f'(1) = 6$

eq. retta tangente in P:

$$y - y_P = m(x - x_P)$$
$$y - 1 = 6(x-1)$$
$$y = 6x - 5$$

$y = e^x \qquad (x_0 = 0)$

$y' = e^x \qquad f(x_0) = 1 \qquad f'(x_0) = 1$

Tangente:

$$y - f(x_0) = f'(x_0)(x - x_0)$$
$$y - 1 = x \quad \rightarrow \quad y = x + 1$$

$$y = \frac{x+2}{x} \qquad (x=-1)$$

$$f(-1) = -1 \qquad P(-1;-1)$$

$$f'(x) = \frac{x-x-2}{x^2} = \frac{-2}{x^2} \qquad m = f'(-1) = -2$$

eq. retta tangente in P:

$$y - y_P = m(x - x_P)$$

$$y + 1 = -2(x+1)$$

$$y = -2x - 3$$

$$y = \log x \qquad (x_0 = e)$$

$$y' = \frac{1}{x} \qquad f(x_0) = 1 \qquad f'(x_0) = \frac{1}{e}$$

tangente:

$$y - f(x_0) = f'(x_0)(x - x_0)$$

$$y - 1 = \frac{1}{e}(x - e) \rightarrow ey - e = x - e \rightarrow x - ey = 0$$

$$y = \frac{4}{7}x - \frac{2}{7}$$

$$y = \sqrt{3x^2-2} \qquad (x=-3)$$

$$f(-3) = 5 \qquad P(-3;5)$$

$$f'(x) = \frac{6x}{2\sqrt{3x^2-2}} = \frac{3x}{\sqrt{3x^2-2}} \qquad m = f'(-3) = -\frac{9}{5}$$

eq. retta tangente in P:

$$y - y_P = m(x - x_P)$$

$$y - 5 = -\frac{9}{5}(x+3)$$

$$y = -\frac{9}{5}x - \frac{2}{5}$$

$$y = x - \cos x \qquad (x_0 = 0)$$

$$y' = 1 + \sin x \qquad f(x_0) = -1 \qquad f'(x_0) = 1$$

tangente:

$$y - f(x_0) = f'(x_0)(x - x_0)$$

$$y + 1 = x \longrightarrow y = x - 1$$

$x^2+y^2=25 \qquad (x_0=3)$

$9+y^2=25$

$y^2=16$

$y=\pm 4$

$P_1\,(3;-4)$

$y=-\sqrt{25-x^2}$

$f'(x)=\dfrac{x}{\sqrt{25-x^2}}$

$f'(3)=\dfrac{3}{4}$

$y+4=\dfrac{3}{4}(x-3)$

$y=\dfrac{3}{4}x-\dfrac{25}{4}$

$P_2\,(3;4)$

$y=\sqrt{25-x^2}$

$f'(x)=-\dfrac{x}{\sqrt{25-x^2}}$

$f'(3)=-\dfrac{3}{4}$

$y-4=-\dfrac{3}{4}(x-3)$

$y=-\dfrac{3}{4}x+\dfrac{25}{4}$

$$y = \operatorname{arctg} \frac{2-\cos x}{1+\cos x} \qquad \left(x = \frac{\pi}{3}\right)$$

$$f\left(\frac{\pi}{3}\right) = \operatorname{arctg} \frac{2-\frac{1}{2}}{1+\frac{1}{2}} = \operatorname{arctg} 1 = \frac{\pi}{4}$$

$$f'(x) = \frac{1}{1+\left(\frac{2-\cos x}{1+\cos x}\right)^2} \cdot \frac{\sin x + \cancel{\sin x \cos x} + 2\sin x - \cancel{\sin x \cos x}}{(1+\cos x)^2}$$

$$f'(x) = \frac{3\sin x}{(1+\cos x)^2 + (2-\cos x)^2} = \frac{3\sin x}{1+2\cos x+\cos^2 x+4-4\cos x+\cos^2 x} = \frac{3\sin x}{2\cos^2 x - 2\cos x + 5}$$

$$f'\left(\frac{\pi}{3}\right) = \frac{\frac{3}{2}\sqrt{3}}{\frac{1}{2} - 1 + 5} = \frac{\cancel{3}^{1}}{\cancel{2}_{1}}\sqrt{3} \cdot \frac{\cancel{2}^{1}}{\cancel{9}_{3}} = \frac{\sqrt{3}}{3}$$

Tangente:

$$y - f\left(\frac{\pi}{3}\right) = f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right)$$

$$y - \frac{\pi}{4} = \frac{\sqrt{3}}{3}\left(x - \frac{\pi}{3}\right)$$

$$y = \frac{\sqrt{3}}{3}x + \frac{\pi}{4} - \frac{\sqrt{3}\,\pi}{9}$$

$$y = \frac{\sqrt{3}}{3}x + \frac{(9-4\sqrt{3})\pi}{36}$$

# Applicazione derivate: fisica

$$s = v_0 t + \frac{1}{2} g t^2$$

$$v = s'(t) = v_0 + g t \qquad a = v'(t) = s''(t) = g$$

$$s = f(t) = \frac{1}{2} g \sin\alpha \cdot t^2$$

$$v = s'(t) = g \sin\alpha \cdot t \qquad a = s''(t) = g \sin\alpha$$

$$v(2) = 9{,}8 \text{ m/s} \qquad \alpha = ?$$

$$9{,}8 \text{ m/s} = 9{,}8 \text{ m/s}^2 \cdot \sin\alpha \cdot 2\text{s} \rightarrow \sin\alpha = \frac{1}{2} \rightarrow \alpha = 30°$$

$$s = 2t^2 + 4t + 4$$

$$v = s'(t) = 4t + 4 \qquad a = s''(t) = 4$$

$$s = 3t^2 + 2t + 2$$

$$v = (6t + 2)\ m/s$$

$$a = 6\ m/s^2$$

$$s = 4 - t$$

$$v = s'(t) = -1\ m/s$$

$$a = s''(t) = 0\ m/s^2$$

$t = 0 \qquad x = 0$

$t = \frac{T}{4} \qquad x = 2\,m \qquad f = \frac{120}{60\,s} = 2\,Hz \qquad T = \frac{1}{2}\,s$

$$\omega = 2\pi f = \frac{2\pi}{T} = 4\pi \ \frac{1}{s}$$

$$x = a \sin(\omega t + \delta)$$

$$\begin{cases} 0 = a \sin\delta \\ 2 = a \sin\left(\frac{2\pi}{T} \cdot \frac{T}{4} + \delta\right) \end{cases} \rightarrow \begin{cases} \delta = 0 \\ 2 = a \sin\frac{\pi}{2} \end{cases} \rightarrow \begin{cases} \delta = 0 \\ a = 2 \end{cases}$$

$$x = 2 \sin 4\pi t$$

$$v = x'(t) = 8\pi \cos 4\pi t$$

$$a = v'(t) = x''(t) = -32\pi^2 \sin 4\pi t = -16\pi^2 x = -\omega^2 x$$

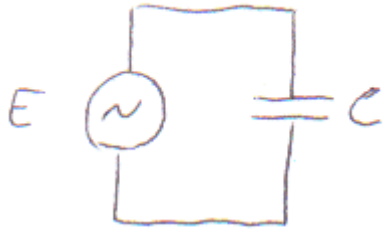
$L_0 = 1\ m \qquad t_0 = 0°C$

$L(t) = (1 + 0{,}000024\,t)\ m \qquad \lambda = ?$

$L(t) = L_0\left(1 + \lambda(t - t_0)\right) \qquad L'(t) = L_0\,\lambda$

$\lambda = \frac{1}{L_0}\frac{dL}{dt} \qquad \lambda = 0{,}000024\ \frac{1}{°C}$

$q = e^{-2t+4} \qquad i = ? \qquad i = \frac{dq}{dt} = -2e^{-2t+4}$




$E = E_0 \sin \omega t$ f.e.m. alternata

$$E - \frac{q}{C} = 0 \rightarrow q = CE \rightarrow q = CE_0 \sin \omega t$$

$$i = \frac{dq}{dt} = \omega C E_0 \cos \omega t$$

$$i = \omega C E_0 \sin\left(\omega t + \frac{\pi}{2}\right)$$

# Teoremi fondamentali del calcolo differenziale

# Teorema di Rolle

$y = x^2 - 4x + 3 \qquad [-1;5]$

la funzione è continua in $[-1;5]$ e derivabile in $(-1;5)$

$f(-1) = 8 \qquad f(5) = 8$

sono soddisfatte le ipotesi del teorema di Rolle

$\exists\, c \in (-1;5) \mid f'(c) = 0$

$f'(x) = 2x - 4 \qquad f'(c) = 0 \rightarrow 2c - 4 = 0 \rightarrow c = 2$

$y = x^4 + x^2 + 1$ $\qquad$ $f$ è continua in $[-2;2]$ e derivabile in $(-2;2)$

$f(-2) = 21 \qquad f(2) = 21 \qquad f'(x) = 4x^3 + 2x$

$\exists\, c \in (-2;2) \mid f'(c) = 0 \qquad 4c^3 + 2c = 0 \rightarrow 2c(2c^2+1) = 0 \rightarrow c = 0$

$y = \cos x + \cos 2x$

$y' = -\sin x - 2\sin 2x$

$f$ è continua in $\left[-\frac{\pi}{4}; \frac{\pi}{4}\right]$ e derivabile in $\left(-\frac{\pi}{4}; \frac{\pi}{4}\right)$

$f\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ $\quad$ $f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$\exists\, c \in \left(-\frac{\pi}{4}; \frac{\pi}{4}\right) \mid f'(c) = 0$

$-\sin c - 2\sin 2c = 0$

$-\sin c - 4\sin c \cos c = 0$

$\sin c\,(1 + 4\cos c) = 0$ $\quad$ $c = 0$ accett.

$c = \pm \arccos\left(-\frac{1}{4}\right)$ NON accettabili

$y = \sin x - \sqrt{3}\cos x + \sqrt{3}$

$y' = \cos x + \sqrt{3}\sin x$

$f$ è continua in $\left[\frac{2}{3}\pi; \pi\right]$ e derivabile in $\left(\frac{2}{3}\pi; \pi\right)$

$f\left(\frac{2}{3}\pi\right) = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \sqrt{3} = 2\sqrt{3}$

$f(\pi) = \sqrt{3} + \sqrt{3} = 2\sqrt{3}$

$\exists\, c \in \left(\frac{2}{3}\pi; \pi\right) \mid f'(c) = 0$

$\cos c + \sqrt{3}\sin c = 0$

$1 + \sqrt{3}\,\mathrm{tg}\, c = 0$

$\mathrm{tg}\, c = -\frac{1}{\sqrt{3}}$ $\quad$ $c = \frac{5}{6}\pi$

$$y = \frac{x}{x^2+4}$$

$f$ è continua in $[1;4]$ e derivabile in $(1;4)$

$$y' = \frac{x^2+4-2x^2}{(x^2+4)^2} = \frac{4-x^2}{(x^2+4)^2}$$

$$f(1) = \frac{1}{5} \qquad f(4) = \frac{4}{20} = \frac{1}{5}$$

$$\exists\, c \in (1;4) \mid f'(c) = 0$$

$$\frac{4-c^2}{(c^2+4)^2} = 0 \rightarrow c = \pm 2 \rightarrow$$

$c_1 = 2$ accett.

$c_2 = -2$ non accett.

$$y = e^{-x^2}$$

$$y' = -2x e^{-x^2}$$

$f$ è continua in $[-3;3]$ e derivabile in $(-3;3)$

$$f(-3) = f(3) = e^{-9}$$

$$\exists\, c \in (-3;3) \mid f'(c) = 0$$

$$-2c e^{-c^2} = 0 \rightarrow c = 0$$

$$y = \frac{1}{2x-1} + \frac{3x}{4x^2-1}$$

$$\lim_{x \to -\frac{1}{2}^-} f(x) = -\infty$$

$$\lim_{x \to -\frac{1}{2}^+} f(x) = +\infty$$

$x = -\frac{1}{2}$ punto di discontinuità di $2^a$ specie

Il teorema di Rolle non è applicabile

$$y = 2x + |5-x| - 7 = \begin{cases} 2x+5-x-7 & \text{se } x<5 \\ 2x+x-5-7 & \text{se } x \geq 5 \end{cases} = \begin{cases} x-2 & \text{se } x<5 \\ 3x-12 & \text{se } x \geq 5 \end{cases}$$

$$y' = \begin{cases} 1 & \text{se } x<5 \\ 3 & \text{se } x>5 \end{cases}$$

$f$ è continua in $[2;5]$ e derivabile in $(2;5)$

$f(2)=0$ $\quad f(5)=3$ $\quad f(2) \neq f(5)$

Il teorema di Rolle non è applicabile.

# Teorema di Lagrange

$y = x^2 - 6x + 9$ in $[3;6]$

$f$ è continua in $[3;6]$

$f$ è derivabile in $(3;6)$

$\exists\, c \in (3;6) \mid f'(c) = \dfrac{f(b) - f(a)}{b - a}$

$f'(x) = 2x - 6$ $\quad f(3) = 9 - 18 + 9 = 0$ $\quad f(6) = 36 - 36 + 9 = 9$

$2c - 6 = \dfrac{9}{3} \rightarrow 2c = 9 \rightarrow c = \dfrac{9}{2}$

$y = x^3 - x^2 + x - 1$ in $[0;2]$

$f$ è continua in $[0;2]$
$f$ è derivabile in $(0;2)$ $\rightarrow \exists\, c \in (0;2) \mid f'(c) = \dfrac{f(2)-f(0)}{2}$

$f'(x) = 3x^2 - 2x + 1$ $\qquad f(2) = 5 \qquad f(0) = -1$

$3c^2 - 2c + 1 = \dfrac{6}{2} \rightarrow 3c^2 - 2c - 2 = 0 \qquad \dfrac{\Delta}{4} = 1 + 6 = 7$

$$c_{1,2} = \frac{1 \pm \sqrt{7}}{3} = \begin{cases} \dfrac{1-\sqrt{7}}{3} \approx -0{,}54858 & \text{non accett.} \\ \dfrac{1+\sqrt{7}}{3} \approx 1{,}215 & \text{accett.} \end{cases}$$

$y = \dfrac{2x}{x+1}$ in $[1;3]$

$D = \mathbb{R} - \{-1\}$

$f$ è continua in $[1;3]$
$f$ è derivabile in $(1;3)$ $\rightarrow \exists\, c \in (1;3) \mid f'(c) = \dfrac{f(3)-f(1)}{3-1}$

$f'(x) = \dfrac{2x+2-2x}{(x+1)^2} = \dfrac{2}{(x+1)^2} \qquad f(3) = \dfrac{6}{4} = \dfrac{3}{2} \qquad f(1) = 1$

$$\frac{2}{(c+1)^2} = \frac{\frac{1}{2}}{2} \rightarrow (c+1)^2 = 8 \rightarrow c+1 = \pm 2\sqrt{2} \rightarrow c = \begin{cases} -1-2\sqrt{2} & \text{non accett.} \\ -1+2\sqrt{2} & \text{accett.} \end{cases}$$

$y = \sqrt{1-x}$ in $[-3;0]$

$D = (-\infty; 1]$

$f$ è continua in $[-3;0]$
$f$ è derivabile in $(-3;0)$ $\longrightarrow$ $\exists\, c \in (-3;0) \mid f'(c) = \dfrac{f(0)-f(-3)}{3}$

$f'(x) = -\dfrac{1}{2\sqrt{1-x}}$ $\qquad f(0) = 1 \qquad f(-3) = 2$

$-\dfrac{1}{2\sqrt{1-c}} = \dfrac{1-2}{3} \rightarrow \sqrt{1-c} = \dfrac{3}{2} \rightarrow 1-c = \dfrac{9}{4} \rightarrow c = -\dfrac{5}{4}$

$y = x^2 + |x|$ in $[-1;2]$

$f$ è continua in $[-1;2]$

$f$ non è derivabile in $x=0$ perché

$$\lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{h^2+|h|}{h} = \lim_{h\to 0}\left(h+\frac{|h|}{h}\right) = \begin{cases} -1 & \text{se } h\to 0^- \\ +1 & \text{se } h\to 0^+ \end{cases}$$

Il teorema di Lagrange non è applicabile.

$y = \frac{2}{x-2}$ in $[0;2]$

$\lim_{x\to 2^-} f(x) = -\infty$

$f$ non è continua in $x=2$, quindi il teorema di Lagrange non è applicabile.

# Teorema di Cauchy

$f(x) = 3x^2 - 1$ $\qquad$ $g(x) = 4x - 3$ $\qquad$ $f$ e $g$ sono continue in $[0;1]$

$f'(x) = 6x$ $\qquad$ $g'(x) = 4$ $\qquad$ $f$ e $g$ sono derivabili in $(0;1)$

$g'(x) \neq 0 \quad \forall x \in (0;1)$

$$\exists\, c \in (0;1) \mid \quad \frac{f'(c)}{g'(c)} = \frac{f(1) - f(0)}{g(1) - g(0)}$$

$$\frac{6c}{4} = \frac{2-(-1)}{1-(-3)} \longrightarrow \frac{6c}{\cancel{4}} = \frac{3}{\cancel{4}} \longrightarrow c = \frac{1}{2}$$

$f(x) = x^3 + 3$ $\qquad$ $g(x) = 2x^2$ $\qquad$ $f$ e $g$ sono continue in $[-1;3]$

$f'(x) = 3x^2$ $\qquad$ $g'(x) = 4x$ $\qquad$ $f$ e $g$ sono derivabili in $(-1;3)$

$g'(0) = 0$ $\qquad$ la terza ipotesi del teorema di Cauchy non è verificata

Il teorema non è applicabile

$f(x) = \sin x + \cos x + 1$ $\qquad$ $g(x) = \sin x - \cos x - 1$ $\qquad$ $f$ e $g$ sono continue in $\left[0;\frac{\pi}{2}\right]$ e

$f'(x) = \cos x - \sin x$ $\qquad$ $g'(x) = \cos x + \sin x$ $\qquad$ derivabili in $\left(0;\frac{\pi}{2}\right)$

$$\exists\, c \in \left(0;\frac{\pi}{2}\right) \mid \frac{f'(c)}{g'(c)} = \frac{f\left(\frac{\pi}{2}\right) - f(0)}{g\left(\frac{\pi}{2}\right) - g(0)}$$

$g'(x) \neq 0 \quad \forall x \in \left(0;\frac{\pi}{2}\right)$

$$\frac{\cos c - \sin c}{\cos c + \sin c} = \frac{2-2}{0-(-2)} \rightarrow \cos c - \sin c = 0$$

$$\operatorname{tg} c = 1 \rightarrow c = \frac{\pi}{4}$$

$f(x) = e^{x^2}$ $\qquad$ $g(x) = 3x^2 + 5$ $\qquad$ $f$ e $g$ sono continue in $[1;2]$ e

$f'(x) = 2x e^{x^2}$ $\qquad$ $g'(x) = 6x$ $\qquad$ derivabili in $(1;2)$

$g'(x) \neq 0 \quad \forall x \in (1;2)$

$$\exists\, c \in (1;2) \;\Big|\; \frac{f'(c)}{g'(c)} = \frac{f(2) - f(1)}{g(2) - g(1)}$$

$$\frac{2c e^{c^2}}{6c} = \frac{e^4 - e}{17 - 8} \longrightarrow e^{c^2} = \frac{e}{3}\left(e^3 - 1\right)$$

$$c^2 = \log \frac{e\left(e^3 - 1\right)}{3} = 1 + \log\left(e^3 - 1\right) - \log 3$$

$$c = \sqrt{1 + \log\left(e^3 - 1\right) - \log 3} \qquad c \simeq 1{,}6883$$

$$f(x) = |1 - \log x| = \begin{cases} 1 - \log x & \text{in } [2;e) \\ \log x - 1 & \text{in } [e;4] \end{cases}$$

$$f'(x) = \begin{cases} -\frac{1}{x} & 2 \leq x < e \\ \frac{1}{x} & e < x \leq 4 \end{cases}$$

La $f$ NON è derivabile in $x = e$ $\rightarrow$ Il teorema di Cauchy NON è applicabile

$g(x) = \log(x-1)$

$g'(x) = \frac{1}{x-1}$

$g$ è continua in $[2;4]$ e derivabile in $(2;4)$

$g'(x) \neq 0 \quad \forall x \in (2;4)$

$$f(x) = |1 - \log x| = \begin{cases} 1 - \log x & \text{in } [2; e) \\ \log x - 1 & \text{in } [e; 4] \end{cases}$$

$$f'(x) = \begin{cases} -\frac{1}{x} & 2 \leq x < e \\ \frac{1}{x} & e < x \leq 4 \end{cases}$$

la $f$ NON è derivabile in $x = e$ $\rightarrow$ Il teorema di Cauchy NON è applicabile

$g(x) = \log(x-1)$

$g'(x) = \frac{1}{x-1}$

$g$ è continua in $[2;4]$ e derivabile in $(2;4)$

$g'(x) \neq 0 \quad \forall x \in (2;4)$

# Teorema di De L'Hopital

$$\lim_{x\to 1} \frac{x^2-1}{x^2-3x+2} = \lim_{x\to 1} \frac{2x}{2x-3} = -2$$

$$\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2} = \lim_{x\to 4} \frac{1}{\frac{1}{2\sqrt{x}}} = 4$$

$$\lim_{x\to 1} \frac{x-1}{\sqrt[3]{x}-1} = \lim_{x\to 1} \frac{1}{\frac{1}{3\sqrt[3]{x^2}}} = 3$$

$$\lim_{x\to 0} \frac{\sin 3x}{\sin 2x} = \lim_{x\to 0} \frac{3\cos 3x}{2\cos 2x} = \frac{3}{2}$$

$$\lim_{x\to 0} \frac{\operatorname{tg} 3x + \operatorname{tg}^3 x}{\operatorname{tg} x} = \lim_{x\to 0} \frac{\frac{3}{\cos^2 3x} + 3\operatorname{tg}^2 x \frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x}} = 3$$

$$\lim_{x\to 0^-} \frac{\operatorname{tg} x}{\cos x - 1} = \lim_{x\to 0^-} \frac{\frac{1}{\cos^2 x}}{-\sin x} = +\infty$$

$$\lim_{x\to 0^-} \frac{\operatorname{tg} x}{\cos x - 1} = \lim_{x\to 0^-} \frac{\frac{1}{\cos^2 x}}{-\sin x} = +\infty$$

$$\lim_{x\to \frac{\pi}{2}} \frac{(1-\sin x)^2}{\cos x} = \lim_{x\to \frac{\pi}{2}} \frac{2(1-\sin x)(-\cos x)}{(-\sin x)} = 0$$

$$\lim_{x\to 0} \frac{e^x - e^{-x}}{\sin x} = \lim_{x\to 0} \frac{e^x + e^{-x}}{\cos x} = 2$$

$$\lim_{x\to \mp\infty} \frac{x^2-3x+1}{1-2x} = \lim_{x\to \mp\infty} \frac{2x-3}{-2} = \pm\infty$$

$$\lim_{x\to +\infty} \frac{e^x}{2x^2+1} = \lim_{x\to +\infty} \frac{e^x}{4x} = \lim_{x\to +\infty} \frac{e^x}{4} = +\infty$$

$$\lim_{x\to 1^+} \frac{\log(x-1)}{e^{\frac{1}{x-1}}} = \lim_{x\to 1^+} \frac{\frac{1}{x-1}}{-\frac{1}{(x-1)^2}\, e^{\frac{1}{x-1}}} = \lim_{x\to 1^+} \frac{1-x}{e^{\frac{1}{x-1}}} = 0$$

$$\lim_{x\to -\infty} \frac{x^2+x+1}{\log(1-x)} = \lim_{x\to -\infty} \frac{2x+1}{\frac{1}{x-1}} = +\infty$$

$$\lim_{x\to \infty} \frac{e^{x^2}}{x^2} = \lim_{x\to \infty} \frac{\cancel{2x}\, e^{x^2}}{\cancel{2x}} = +\infty$$

$$\lim_{x\to 0^+} (x\log x) = \lim_{x\to 0^+} \frac{\log x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} (-x) = 0$$

$$\lim_{x\to+\infty}\left[e^{-x}(x^2-1)\right]=\lim_{x\to+\infty}\frac{x^2-1}{e^x}=\lim_{x\to+\infty}\frac{2x}{e^x}=\lim_{x\to+\infty}\frac{2}{e^x}=0$$

$$\lim_{x\to0^+}\left(x^3\log x\right)=\lim_{x\to0^+}\frac{\log x}{\frac{1}{x^3}}=\lim_{x\to0^+}\frac{\frac{1}{x}}{\frac{-3}{x^4}}=\lim_{x\to0^+}\left(\frac{-x^3}{3}\right)=0$$

$$\lim_{x\to1^+}\left[(x^2-1)\log(x-1)\right]=\lim_{x\to1^+}\frac{\log(x-1)}{\frac{1}{x^2-1}}=\lim_{x\to1^+}\frac{\frac{1}{x-1}}{-\frac{2x}{(x^2-1)^2}}=\lim_{x\to1^+}\frac{(x-1)^{\cancel{2}}(x+1)^2}{-2x\cancel{(x-1)}}=0$$

$$\lim_{x\to\infty}\left[\left(1-e^{\frac{1}{x}}\right)\cdot x\right]=\lim_{x\to\infty}\frac{1-e^{\frac{1}{x}}}{\frac{1}{x}}=\lim_{x\to\infty}\frac{\cancel{\frac{1}{x^2}}\,e^{\frac{1}{x}}}{-\cancel{\frac{1}{x^2}}}=-1$$

$$\lim_{x\to1^\pm}x^{\frac{1}{x-1}}=\lim_{x\to1^\pm}e^{\frac{\log x}{x-1}}=e^{\lim_{x\to1^\pm}\frac{\frac{1}{x}}{1}}=e$$

$$\lim_{x\to 1^+} (\log x)^{1-x} = \lim_{x\to 1^+} e^{(1-x)\log\log x} = \lim_{x\to 1^+} e^{\frac{\log\log x}{\frac{1}{1-x}}} = e^{\lim_{x\to 1^+} \frac{\frac{1}{x\log x}}{\frac{1}{(1-x)^2}}} =$$

$$= e^{\lim_{x\to 1^+} \frac{1-2x+x^2}{x\log x}} = e^{\lim_{x\to 1^+} \frac{-2+2x}{\log x+1}} = e^0 = 1$$

$$\lim_{x\to 0^+} (\sin x)^{\operatorname{tg} x} = e^{\lim_{x\to 0^+} \operatorname{tg} x \log\sin x} = e^{\lim_{x\to 0^+} \frac{\log\sin x}{\operatorname{cotg} x}} = e^{\lim_{x\to 0^+} \frac{\frac{\cos x}{\sin x}}{-\frac{1}{\sin^2 x}}} =$$

$$= e^{\lim_{x\to 0^+} (-\sin x\cos x)} = e^0 = 1$$

$$\lim_{x\to 0^+} (\sin x\cdot\log x) = \lim_{x\to 0^+} \frac{\log x}{\frac{1}{\sin x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{\cos x}{\sin^2 x}} = \lim_{x\to 0^+} \frac{-\sin^2 x}{x\cos x} =$$

$$= \lim_{x\to 0^+} \frac{-2\sin x\cos x}{\cos x - x\sin x} = 0$$

# Problemi di massimo e di minimo

# Problemi di geometria

(1) $S = 2\pi r^2 + 2\pi r h$ $\qquad$ $S$ costante

cilindro di volume massimo ?

(2) $V = \pi r^2 h$

ricavo $h$ dalla (1) e sostituisco nella (2)

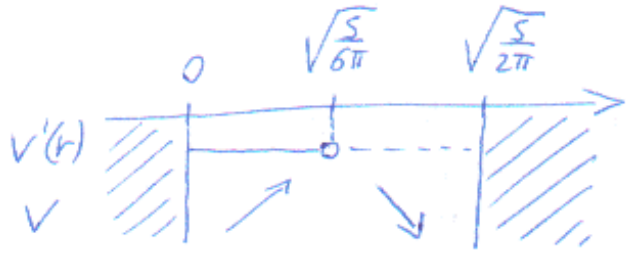
$$S - 2\pi r^2 = 2\pi r h \quad \rightarrow \quad h = \frac{S - 2\pi r^2}{2\pi r}$$

$$V = \pi r^2 \cdot \frac{S - 2\pi r^2}{2\pi r} = \frac{1}{2} S r - \pi r^3$$

studio la funzione $V = V(r) = \frac{1}{2} S r - \pi r^3$ $\qquad$ con $0 < r < \sqrt{\frac{S}{2\pi}}$

$$\frac{dV}{dr} = \frac{1}{2}S - 3\pi r^2$$

$$\frac{dV}{dr} > 0 \rightarrow \frac{1}{2}S - 3\pi r^2 > 0 \rightarrow S > 6\pi r^2 \rightarrow r^2 < \frac{S}{6\pi}$$

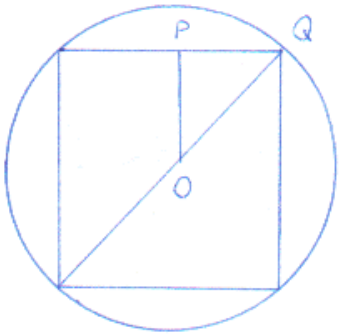



$r = \sqrt{\frac{S}{6\pi}}$ punto di max.

$$h = \frac{S - 2\pi \frac{S}{6\pi}}{2\pi\sqrt{\frac{S}{6\pi}}} = \frac{S - \frac{1}{3}S}{\sqrt{\frac{2}{3}\pi S}} = \frac{\frac{2}{3}S}{\sqrt{\frac{2}{3}\pi S}} = \sqrt{\frac{2S}{3\pi}} = 2\sqrt{\frac{S}{6\pi}} = 2r$$

Quando il volume è massimo, l'altezza del cilindro è il doppio del raggio di base, pertanto il cilindro di volume massimo è equilatero.

$$V_{max} = \cancel{\pi}\,\frac{S}{6\cancel{\pi}}\sqrt{\frac{2S}{3\pi}} = \frac{1}{6}S\sqrt{\frac{2S}{3\pi}}$$




$\overline{OQ} = r$ raggio della sfera circoscritta al cilindro

$\overline{PQ} = x$ raggio di base del cilindro

$\overline{OP} = \frac{h}{2}$ h altezza del cilindro

cilindro di volume massimo?

$$\frac{h}{2} = \sqrt{r^2 - x^2} \qquad \text{(teorema di Pitagora)}$$

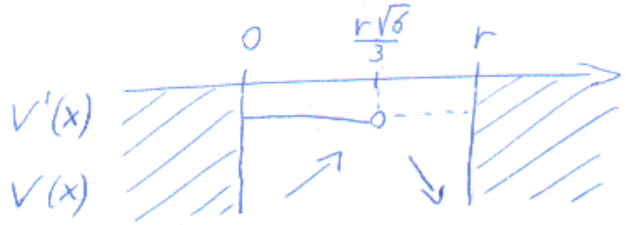
$$h = 2\sqrt{r^2 - x^2} \qquad 0 < x < r$$

$$V = \pi x^2 h = 2\pi x^2 \sqrt{r^2 - x^2} = 2\pi \sqrt{r^2 x^4 - x^6}$$

$$\frac{dV}{dx} = \cancel{2}\pi \frac{4r^2x^3 - 6x^5}{\cancel{2}\sqrt{r^2x^4 - x^6}} = \pi \frac{2x^3(2r^2 - 3x^2)}{\sqrt{r^2x^4 - x^6}}$$

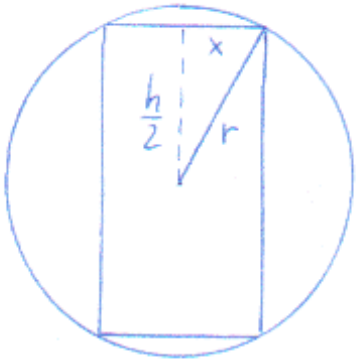
$$\frac{dV}{dx} = 2\pi \frac{x(2r^2 - 3x^2)}{\sqrt{r^2 - x^2}}$$

$$\frac{dV}{dx} > 0 \quad \rightarrow \quad 2r^2 - 3x^2 > 0$$




$x = \frac{r\sqrt{6}}{3}$ punto di max.

Il volume è massimo quando il raggio di base misura $\frac{r\sqrt{6}}{3}$.




$r$ raggio della sfera

cilindro di superficie laterale massima?

$$\frac{h}{2} = \sqrt{r^2 - x^2}$$

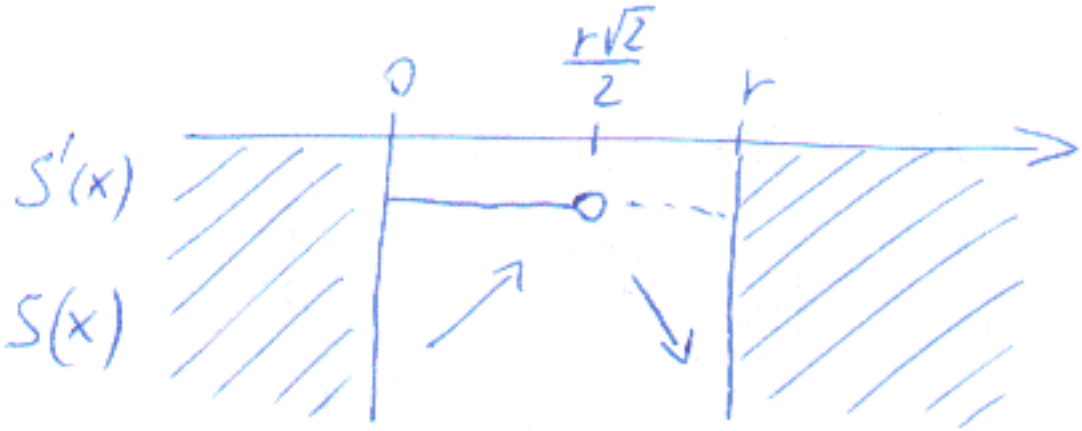
$$h = 2\sqrt{r^2 - x^2} \qquad 0 < x < r$$

$$S = 2\pi x h = 2\pi x \cdot 2\sqrt{r^2 - x^2} = 4\pi\sqrt{r^2x^2 - x^4}$$

$$\frac{dS}{dx} = 4\pi \frac{2r^2x - 4x^3}{2\sqrt{r^2x^2 - x^4}} = 4\pi \cdot \frac{\cancel{2x}(r^2 - 2x^2)}{\cancel{2x}\sqrt{r^2 - x^2}}$$

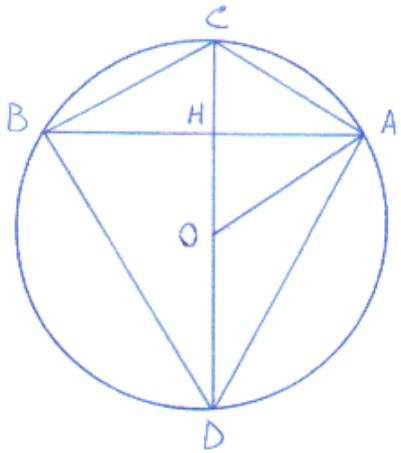
$$\frac{dS}{dx} = \frac{4\pi(r^2 - 2x^2)}{\sqrt{r^2 - x^2}}$$

$$\frac{dS}{dx} > 0 \quad \rightarrow \quad r^2 - 2x^2 > 0$$




$x = \frac{r\sqrt{2}}{2}$ punto di max.

Il cilindro di superficie laterale massima è quello il cui raggio di base misura $\frac{r\sqrt{2}}{2}$.




$\overline{OA} = r$ raggio della sfera

differenza massima dei volumi dei due coni ?

$\overline{OH} = x$ $\qquad 0 < x < r$

$\overline{DH} = r + x$

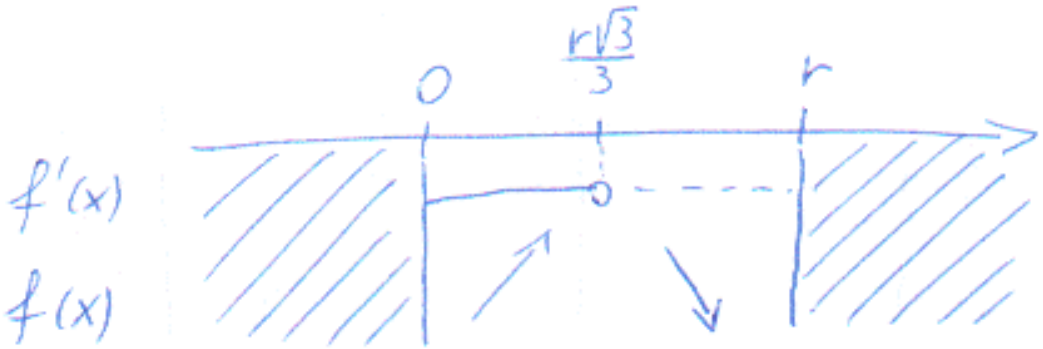
$\overline{CH} = r - x$

$\overline{HA}^2 = \overline{CH} \cdot \overline{DH} = r^2 - x^2$ (2° teorema di Euclide)

$$f(x) = \frac{1}{3}\pi\, \overline{HA}^2 \cdot \overline{DH} - \frac{1}{3}\pi\, \overline{HA}^2 \cdot \overline{CH}$$

$$f(x) = \frac{\pi}{3}\left(r^2 - x^2\right)\left(r + x - r + x\right) = \frac{2\pi}{3}\left(r^2 x - x^3\right)$$

$$f'(x) = \frac{2\pi}{3}\left(r^2 - 3x^2\right)$$




$x = \frac{r\sqrt{3}}{3}$ punto di massimo

La differenza dei volumi dei due coni è massima quando la distanza della base comune dal centro della sfera è $\frac{r\sqrt{3}}{3}$.

$V = \frac{1}{3}\pi a^3$ volume costante

superficie laterale minima?

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{1}{3}\pi a^3 = \frac{1}{3}\pi r^2 h \rightarrow a^3 = r^2 h \rightarrow h = \frac{a^3}{r^2}$$

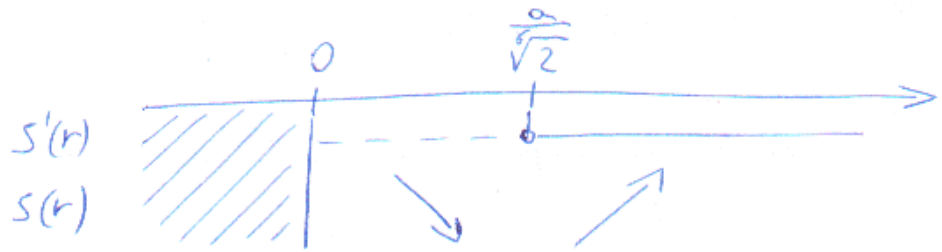
$$\ell = \sqrt{r^2 + h^2} = \sqrt{r^2 + \frac{a^6}{r^4}}$$

$$S = \pi r \ell = \pi r \sqrt{r^2 + \frac{a^6}{r^4}} = \pi r \sqrt{\frac{r^6 + a^6}{r^4}} = \pi \frac{\sqrt{r^6 + a^6}}{r} \qquad r > 0$$

$$\frac{dS}{dr} = \pi \frac{\frac{6r^6}{2\sqrt{r^6+a^6}} - \sqrt{r^6+a^6}}{r^2} = \pi \frac{3r^6 - r^6 - a^6}{r^2\sqrt{r^6+a^6}}$$

$$\frac{dS}{dr} = \pi \frac{2r^6 - a^6}{r^2\sqrt{r^6+a^6}}$$

$$\frac{dS}{dr} > 0 \quad \rightarrow \quad 2r^6 - a^6 > 0$$




$r = \frac{a}{\sqrt[6]{2}}$ punto di minimo

Il cono di superficie laterale minima è quello il cui raggio di base misura $\frac{a}{\sqrt[6]{2}}$.

$\ell$ apotema costante

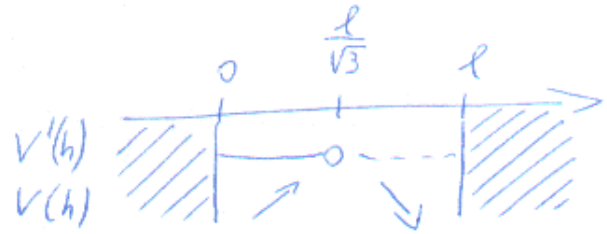
cono di volume massimo?

$$r^2 = \ell^2 - h^2 \qquad 0 < h < \ell$$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(\ell^2 - h^2)h = \frac{\pi}{3}(\ell^2 h - h^3)$$

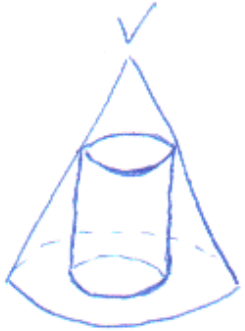
$$\frac{dV}{dh} = \frac{\pi}{3}(\ell^2 - 3h^2)$$

$$\frac{dV}{dh} > 0 \rightarrow \ell^2 - 3h^2 > 0$$



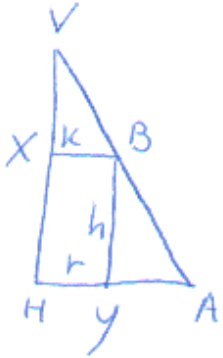

$h = \frac{\ell\sqrt{3}}{3}$ punto di massimo

Il cono di volume massimo è quello la cui altezza è $\frac{\ell\sqrt{3}}{3}$.




cilindro assegnato

cono circoscritto di volume minimo ?




$h$ altezza del cilindro

$r$ raggio di base del cilindro

$x$ altezza del cono circoscritto al cilindro

$y$ raggio di base del cono

$r : y = (x-h) : x$ $\qquad \overset{\triangle}{VKB} \sim \overset{\triangle}{VHA}$

$y(x-h) = rx \rightarrow y = \dfrac{rx}{x-h}$ $\qquad x > h$

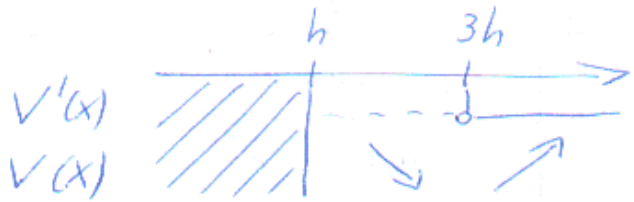
$$V = \frac{1}{3}\pi y^2 x = \frac{\pi}{3}\frac{r^2x^2}{(x-h)^2}\cdot x = \frac{\pi}{3}\frac{r^2x^3}{(x-h)^2}$$

$$\frac{dV}{dx} = \frac{\pi}{3} r^2 \frac{3x^2(x-h)^2 - 2(x-h)x^3}{(x-h)^4}$$

$$\frac{dV}{dx} = \frac{\pi}{3} r^2 x^2 \frac{3(x-h) - 2x}{(x-h)^3} = \frac{\pi r^2 x^2}{3(x-h)^3}\left(3x - 3h - 2x\right)$$

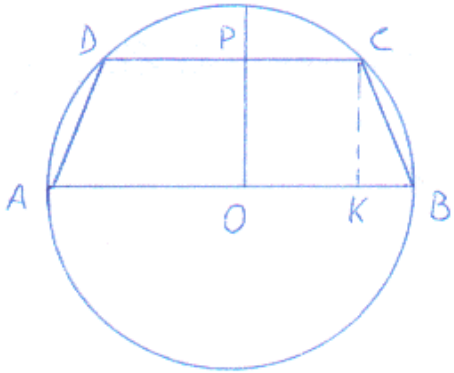
$$\frac{dV}{dx} = \frac{\pi r^2 x^2}{3(x-h)^3}\left(x - 3h\right)$$

$$\frac{dV}{dx} > 0 \quad \rightarrow \quad x > 3h$$




$x = 3h$ punto di minimo

Il cono di volume minimo è quello la cui altezza è tripla di quella del cilindro.




$\overline{OA} = r$ raggio della sfera

tronco di cono inscritto nella sfera di superficie laterale massima?

$\overline{PC} = x$ raggio della base minore del tronco di cono

$0 < x < r$

$\overline{KB} = r - x$

$\overline{CK}^2 = r^2 - x^2$

$\overline{BC}^2 = \overline{KB}^2 + \overline{CK}^2 = r^2 - 2rx + x^2 + r^2 - x^2 = 2r^2 - 2rx = 2r(r-x)$

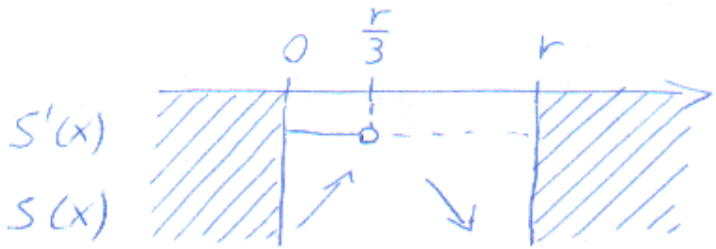
$\overline{BC} = \sqrt{2r(r-x)}$

$$S = \pi\left(\overline{OB} + \overline{PC}\right) \cdot \overline{BC} = \pi\,(r+x) \cdot \sqrt{2r(r-x)}$$

$$\frac{dS}{dx} = \pi\sqrt{2r(r-x)} + \pi(r+x)\frac{(-r)}{\sqrt{2r(r-x)}} = \pi\,\frac{2r^2-2rx-r^2-rx}{\sqrt{2r(r-x)}}$$

$$\frac{dS}{dx} = \pi r\,\frac{r-3x}{\sqrt{2r(r-x)}}$$

$$\frac{dS}{dx} > 0 \quad \rightarrow \quad r-3x>0 \quad \rightarrow \quad x<\frac{r}{3}$$




$x = \frac{r}{3}$ punto di massimo

Il tronco di cono di superficie laterale massima ha il raggio della base minore uguale al $\frac{1}{3}$ del raggio della sfera.

# Problemi di geometria analitica

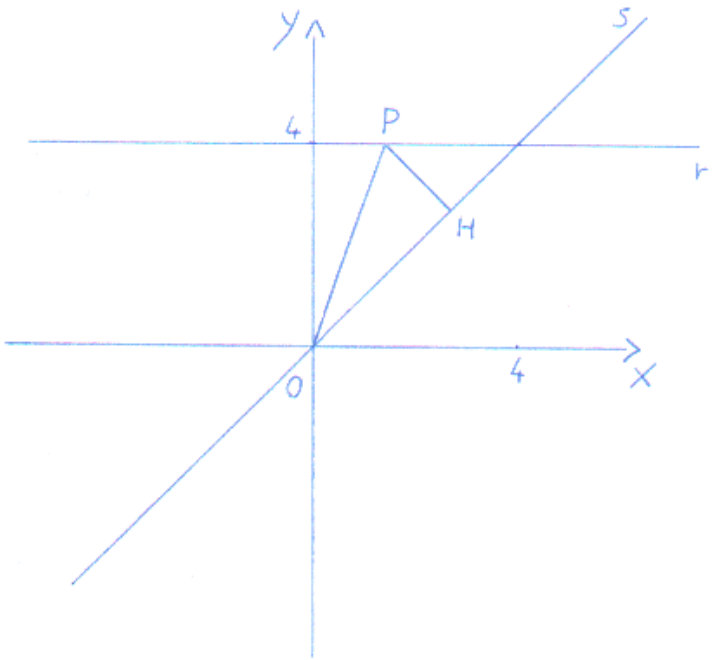
$r: \quad y=4$

$s: \quad y=x$

$P \in r \qquad P(x;4)$

$H$ piede della perpendicolare condotta da $P$ alla retta $s$

$\overline{PO}^2 + \overline{PH}^2$ min. ?




$$\overline{PO}^2 = x^2+16$$

$$\overline{PH} = d(P;s) = \frac{|x-4|}{\sqrt{2}}$$

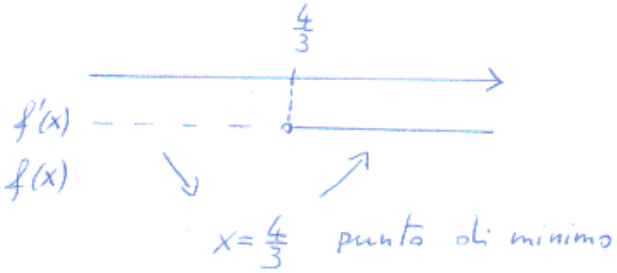
$$\overline{PH}^2 = \frac{1}{2}\left(x^2-8x+16\right)$$

$$f(x) = \overline{PO}^2 + \overline{PH}^2 = x^2+16+\frac{1}{2}x^2-4x+8$$

$$f(x) = \frac{3}{2}x^2-4x+24$$

$$f'(x) = 3x-4$$

$$f'(x)>0 \quad \rightarrow \quad x>\frac{4}{3}$$




Il punto $P$ ha coordinate $\left(\frac{4}{3};4\right)$.

$A(5;2)$

$r: \quad y = 2x + 4$

$P(a;\ 2a+4) \in r$

$a = ?$ $\qquad \overline{PA}$ min.

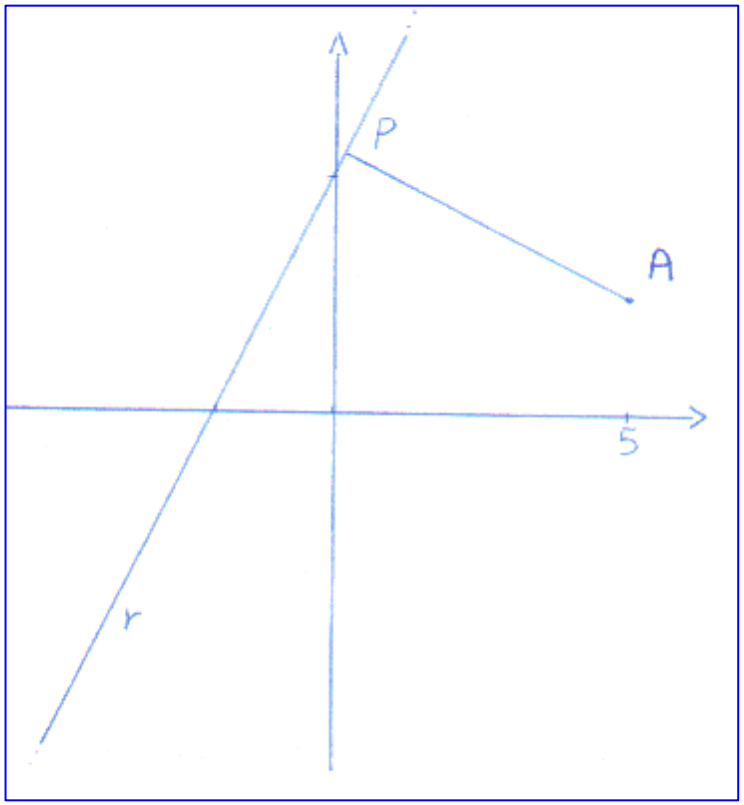
$$\overline{PA}^2 = (5-a)^2 + (2a+2)^2$$

$$\overline{PA}^2 = 25 - 10a + a^2 + 4a^2 + 8a + 4$$

$$\overline{PA}^2 = 5a^2 - 2a + 29$$

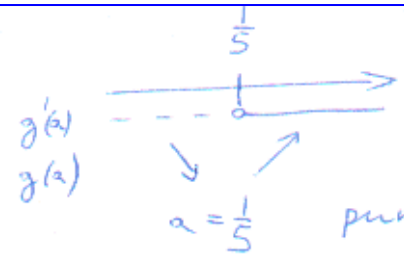
$$f(a) = \overline{PA} = \sqrt{5a^2 - 2a + 29}$$

Poiché $f(a)$ è una funzione positiva $\forall a \in \mathbb{R}$, le funzioni $f(a)$ e $g(a) = f^2(a)$ assumono i valori massimi e minimi per gli stessi valori di $a$.



$g(a) = 5a^2 - 2a + 29$

$g'(a) = 10a - 2 \qquad g'(a) > 0 \rightarrow a > \frac{1}{5}$




punto di minimo

$P\left(\frac{1}{5} ; \frac{22}{5}\right)$

retta $\perp$ ad r passante per A:

$$y - 2 = -\frac{1}{2}(x - 5)$$

$$y = -\frac{1}{2}x + \frac{9}{2}$$

$$H \quad \begin{cases} y = 2x + 4 \\ y = -\frac{1}{2}x + \frac{9}{2} \end{cases} \rightarrow \begin{cases} y = 2x + 4 \\ 2x + 4 = -\frac{1}{2}x + \frac{9}{2} \end{cases} \rightarrow \begin{cases} y = 2x + 4 \\ \frac{5}{2}x = \frac{1}{2} \end{cases} \rightarrow \begin{cases} y = \frac{22}{5} \\ x = \frac{1}{5} \end{cases}$$

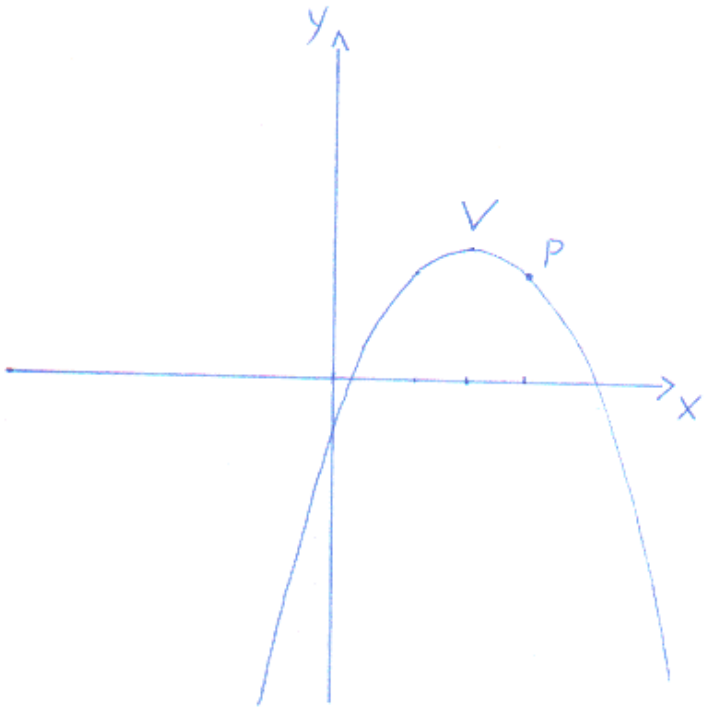
$H\left(\frac{1}{5} ; \frac{22}{5}\right)$

Il piede della perpendicolare da A ad r coincide con P.

parabola $4y + x^2 = 10x - 5$

$P(x;y)$ punto della parabola

$x+y$ max. ?



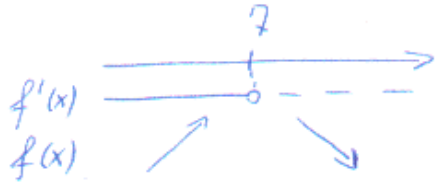

$$y = -\frac{1}{4}x^2 + \frac{5}{2}x - \frac{5}{4}$$

$$f(x) = x - \frac{1}{4}x^2 + \frac{5}{2}x - \frac{5}{4}$$

$$f(x) = -\frac{1}{4}x^2 + \frac{7}{2}x - \frac{5}{4}$$

$$f'(x) = -\frac{1}{2}x + \frac{7}{2}$$

$f'(x) > 0 \rightarrow -x + 7 > 0 \rightarrow x < 7$



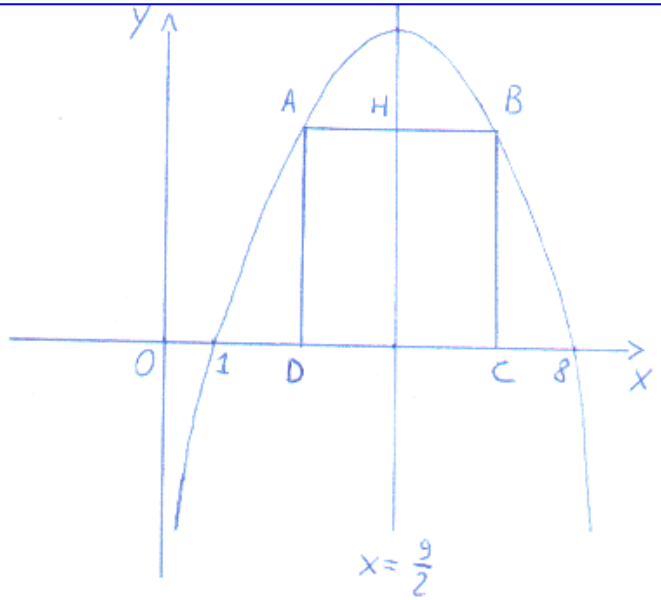

$x = 7$ punto di max.

Il punto della parabola ha coordinate $P(7;4)$

$y = -x^2 + 9x - 8$

asse x : $y = 0$

rettangolo di perimetro massimo ?




$A\left(x; -x^2+9x-8\right) \qquad 1 < x < \frac{9}{2}$

$\overline{AH} = \frac{9}{2} - x \qquad \overline{AB} = 2\,\overline{AH} = 9 - 2x$

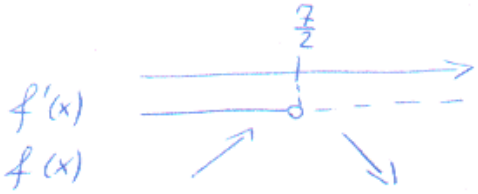
$\overline{AD} = -x^2 + 9x - 8$

$f(x) = 2p = 2\,\overline{AD} + 2\,\overline{AB} = -2x^2 + 18x - 16 + 18 - 4x$

$f(x) = -2x^2 + 14x + 2$

$f'(x) = -4x + 14$

$f'(x) > 0 \rightarrow -4x + 14 > 0 \rightarrow 4x < 14 \rightarrow x < \frac{7}{2}$




$x = \frac{7}{2}$ punto di massimo

$2p = f\left(\frac{7}{2}\right) = -2 \cdot \frac{49}{4} + 14 \cdot \frac{7}{2} + 2 = \frac{53}{2}$

$y = x^2 - 4x + 3$

$y = -x^2 + 6x - 5$

$\overline{PQ}$ massimo

$P(x;\ x^2 - 4x + 3)$

$Q(x;\ -x^2 + 6x - 5)$ $\qquad 1 < x < 4$

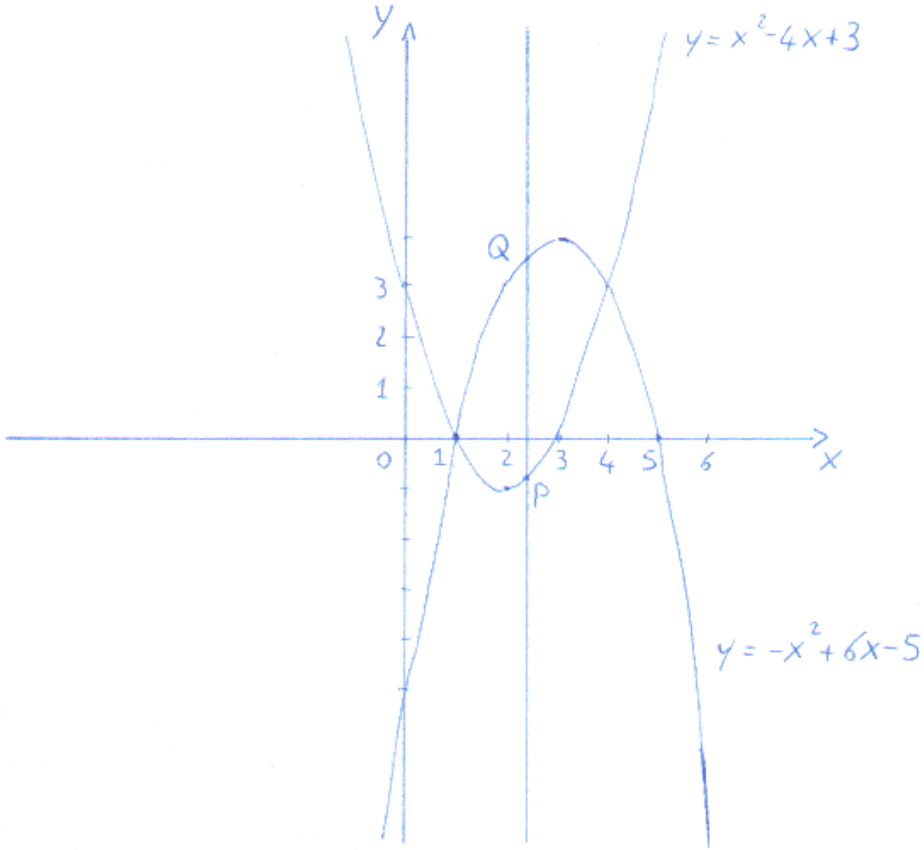
$y_Q > y_P$

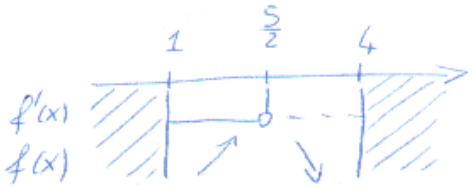
$\overline{PQ} = f(x) = y_Q - y_P = -x^2 + 6x - 5 - (x^2 - 4x + 3)$

$f(x) = -2x^2 + 10x - 8$

$f'(x) = -4x + 10$

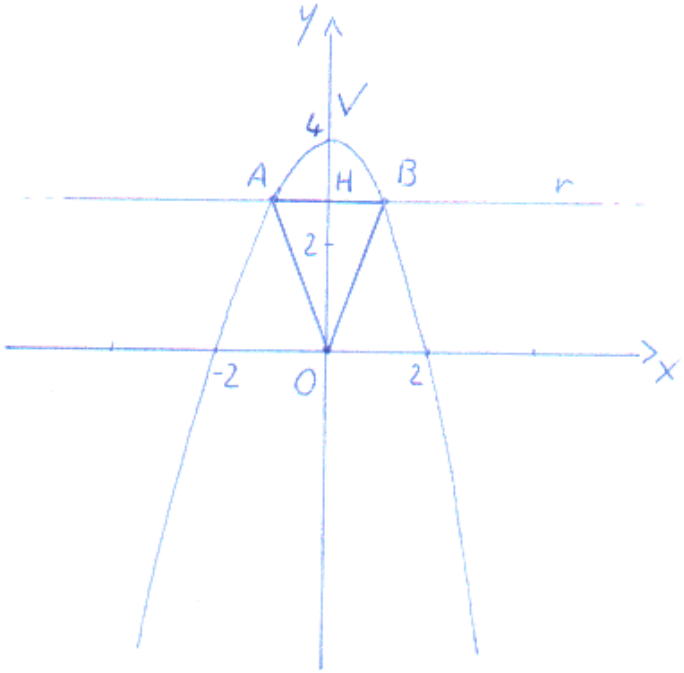
$f'(x) > 0 \rightarrow -4x + 10 > 0 \rightarrow x < \frac{5}{2}$




$x = \frac{5}{2}$ punto di max

$y = 4 - x^2$ area triangolo AOB max ?




$B\,(x\,;\,-x^2+4)$ $\quad 0 < x < 2$

$\overline{BH} = x$

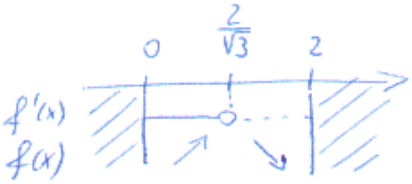
$\overline{AB} = 2x$

$\overline{OH} = y_B = -x^2+4$

$S = \frac{1}{2}\,\overline{AB}\cdot\overline{OH} = f(x) = -x^3+4x$

$f'(x) = -3x^2+4$

$f'(x) > 0 \quad \rightarrow \quad -3x^2+4>0$



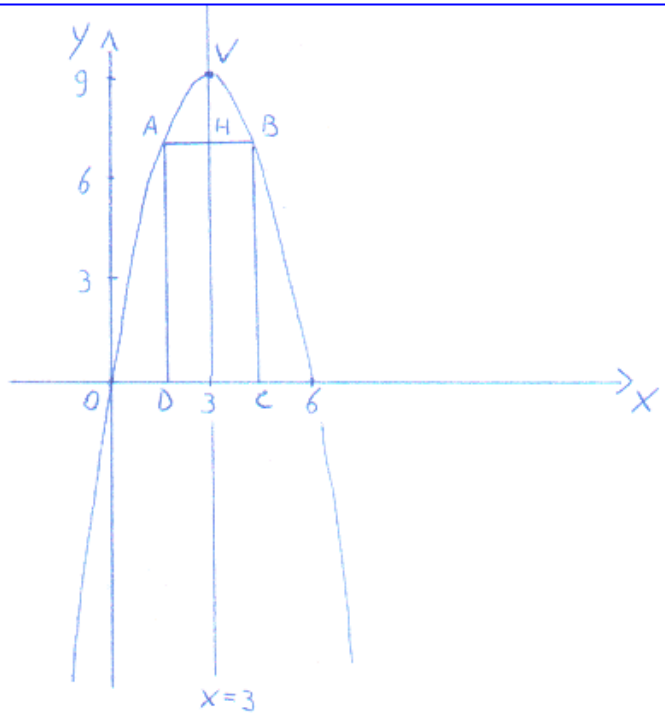

$x = \frac{2\sqrt{3}}{3}$ punto di max

$y = -\frac{4}{3} + 4 = \frac{8}{3}$

$\text{Area} = xy = \frac{16}{9}\sqrt{3}$

$y = -x^2 + 6x$

rettangolo ABCD di area massima?




$x = 3$ asse di simmetria della parabola

$A(x; -x^2+6x)$ $\quad 0 < x < 3$

$\overline{AH} = 3 - x$

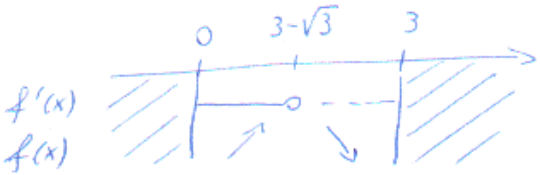
$\overline{AB} = 2\,\overline{AH} = 6 - 2x$

$\overline{AD} = -x^2 + 6x$

$$S = \overline{AD} \cdot \overline{AB} = f(x) = (-x^2+6x)(-2x+6) = 2x^3 - 6x^2 - 12x^2 + 36x = 2x^3 - 18x^2 + 36x$$

$$f'(x) = 6x^2 - 36x + 36 = 6(x^2 - 6x + 6)$$

$f'(x) > 0 \rightarrow x^2 - 6x + 6 > 0 \qquad x_{1,2} = 3 \pm \sqrt{3}$




$x = 3 - \sqrt{3}$ punto di max

$$S = (-9 - 3 \cancel{+6\sqrt{3}} + 18 \cancel{-6\sqrt{3}})(\cancel{-6} + 2\sqrt{3} \cancel{+6}) = 12\sqrt{3}$$

# Massimi, minimi e flessi di funzioni contenenti un parametro

$y = ax^2 + bx$ min in $\left(-\frac{1}{3}; -\frac{1}{3}\right)$

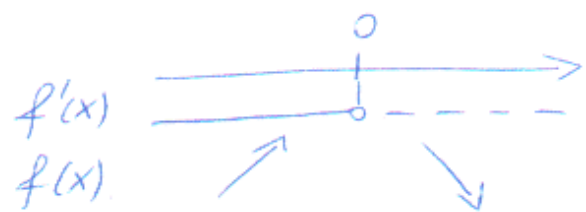
$y' = 2ax + b$

$$\begin{cases} f'\left(-\frac{1}{3}\right) = 0 \\ f\left(-\frac{1}{3}\right) = -\frac{1}{3} \end{cases} \rightarrow \begin{cases} -\frac{2}{3}a + b = 0 \\ \frac{1}{9}a - \frac{1}{3}b = -\frac{1}{3} \end{cases} \rightarrow \begin{cases} -2a + 3b = 0 \\ a - 3b = -3 \end{cases} \rightarrow \begin{cases} -6b + 6 + 3b = 0 \\ a = 3b - 3 \end{cases} \rightarrow$$

$$\rightarrow \begin{cases} 3b = 6 \\ a = 3b - 3 \end{cases} \rightarrow \begin{cases} b = 2 \\ a = 3 \end{cases} \qquad y = 3x^2 + 2x$$

$y = \dfrac{a}{a + x^2}$ max in $(0; 1)$ $a > 0$

$y' = \dfrac{-2ax}{(a + x^2)^2}$ $y' > 0 \rightarrow -2ax > 0 \rightarrow x < 0$




$x = 0$ punto di max

$f(0) = \frac{a}{a} = 1$

$y = -x^2 + bx + c$ max in $(1;4)$

$y' = -2x + b$

$$\begin{cases} f'(1) = 0 \\ f(1) = 4 \end{cases} \rightarrow \begin{cases} -2 + b = 0 \\ -1 + b + c = 4 \end{cases} \rightarrow \begin{cases} b = 2 \\ c = 3 \end{cases} \qquad y = -x^2 + 2x + 3$$

$y = a \sin x + b \cos x$ max in $\left(\frac{\pi}{4}; \sqrt{2}\right)$

$y' = a \cos x - b \sin x$

$$\begin{cases} f'\left(\frac{\pi}{4}\right) = 0 \\ f\left(\frac{\pi}{4}\right) = \sqrt{2} \end{cases} \rightarrow \begin{cases} \frac{\sqrt{2}}{2}(a - b) = 0 \\ \frac{\sqrt{2}}{2}(a + b) = \sqrt{2} \end{cases} \rightarrow \begin{cases} a = b \\ 2a = 2 \end{cases} \rightarrow \begin{cases} b = 1 \\ a = 1 \end{cases}$$

$y = \sin x + \cos x \qquad f\left(\frac{\pi}{4}\right) = \sqrt{2}$

$y' = \cos x - \sin x \qquad f'\left(\frac{\pi}{4}\right) = 0$

$y'' = -\sin x - \cos x \qquad f''\left(\frac{\pi}{4}\right) = -\sqrt{2}$

$x = \frac{\pi}{4}$ punto di max.

$$y = \frac{2x(x-m)}{x^2+x+6} = \frac{2x^2-2mx}{x^2+x+6}$$

$$y' = \frac{(4x-2m)(x^2+x+6)-(2x+1)(2x^2-2mx)}{(x^2+x+6)^2}$$

$$y' = \frac{\cancel{4x^3}+4x^2+24x-2mx^2\cancel{-2mx}-12m\cancel{-4x^3}+4mx^2-2x^2\cancel{+2mx}}{(x^2+x+6)^2}$$

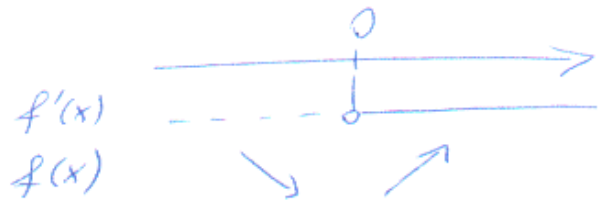
$$y' = \frac{x^2(2+2m)+24x-12m}{(x^2+x+6)^2}$$

$$y'=0 \rightarrow (m+1)x^2+12x-6m=0$$

$$\frac{\Delta}{4}>0 \rightarrow 36+6m^2+6m>0 \rightarrow m^2+m+6>0 \qquad \forall m \in \mathbb{R}$$

La funzione ha dei massimi o dei minimi per m qualsiasi.

$$y = \frac{a}{x^2+1} \qquad y' = -\frac{2ax}{(x^2+1)^2} \qquad y'>0 \rightarrow -2ax>0 \rightarrow \begin{cases} x>0 & \text{se } a<0 \\ x<0 & \text{se } a>0 \end{cases}$$




$x=0$ punto di minimo se $a<0$

$$y = x^3 - ax^2 + 3x + 4a$$

$$y' = 3x^2 - 2ax + 3$$

$$\Delta \leq 0 \rightarrow a^2 - 9 \leq 0 \rightarrow -3 \leq a \leq 3$$

La funzione non presenta né massimi né minimi per $-3 \leq a \leq 3$.

$$y = (a-3)\cos x + b\sin x$$
$$y' = (a-3)(-\sin x) + b\cos x$$
$$y'' = (3-a)\cos x - b\sin x$$

① max in $\left(\frac{\pi}{2}; 2\right)$

$$\begin{cases} 3-a=0 \\ b=2 \end{cases} \rightarrow \begin{cases} a=3 \\ b=2 \end{cases}$$

② min in $\left(\frac{\pi}{4}; -4\sqrt{2}\right)$

$$\begin{cases} \frac{\sqrt{2}}{2}(3-a+b)=0 \\ \frac{\sqrt{2}}{2}(a-3+b)=-4\sqrt{2} \end{cases} \rightarrow \begin{cases} b=a-3 \\ a-3=-4 \end{cases} \rightarrow \begin{cases} b=-4 \\ a=-1 \end{cases}$$

③ flesso con tangente orizzontale in $\left(\frac{\pi}{3}; 0\right)$

$$\begin{cases} \frac{\sqrt{3}}{2}(3-a)+\frac{1}{2}b=0 \\ \frac{1}{2}(a-3)+\frac{\sqrt{3}}{2}b=0 \end{cases} \rightarrow \begin{cases} \frac{3}{2}b+\frac{1}{2}b=0 \\ a=3-\sqrt{3}b \end{cases} \rightarrow$$

$$\rightarrow \begin{cases} b=0 \\ a=3 \end{cases}$$

$y = ax^3 - 5x^2 + 3x + b$ flesso in $A(3;-1)$

$y' = 3ax^2 - 10x + 3$ eq. tangente in $A$

$y'' = 6ax - 10$

$$\begin{cases} f''(3) = 0 \\ f(3) = -1 \end{cases} \rightarrow \begin{cases} 18a - 10 = 0 \\ 27a - 45 + 9 + b = -1 \end{cases} \rightarrow$$

$$\rightarrow \begin{cases} a = \frac{5}{9} \\ b = 20 \end{cases}$$

$$m = f'(3) = 3 \cdot \frac{5}{\cancel{9}_{1}} \cdot \cancel{9}^{1} - 10 \cdot 3 + 3 = -12$$

$$y - y_A = m(x - x_A) \rightarrow y + 1 = -12x + 36 \rightarrow 12x + y - 35 = 0$$

$y = mx^3 + nx^2 + px + m$

flesso in $(1;-2)$ con tangente inclinata di $135°$ sull'asse $x$

$y' = 3mx^2 + 2nx + p$

$m = tg\, 135° = -1$

$y'' = 6mx + 2n$

$$\begin{cases} f''(1) = 0 \\ f(1) = -2 \\ f'(1) = -1 \end{cases} \rightarrow \begin{cases} 6m + 2n = 0 \\ m + n + p + m = -2 \\ 3m + 2n + p = -1 \end{cases} \rightarrow$$

$$\rightarrow \begin{cases} n = -3m \\ 2m - 3m + p = -2 \\ 3m - 6m + p = -1 \end{cases} \rightarrow \begin{cases} n = -3m \\ p = m - 2 \\ -3m + m - 2 = -1 \end{cases} \rightarrow \begin{cases} n = -3m \\ p = m - 2 \\ -2m = 1 \end{cases} \rightarrow \begin{cases} n = \frac{3}{2} \\ p = -\frac{5}{2} \\ m = -\frac{1}{2} \end{cases}$$

# Studio funzione

$$\boxed{y = 4x^3 + 2x^2}$$

dominio:

$D = \mathbb{R}$

simmetrie:

$f(-x) = -4x^3 + 2x^2$

$f(-x) \neq f(x)$ $f$ non è pari

$f(-x) \neq -f(x)$ $f$ non è dispari

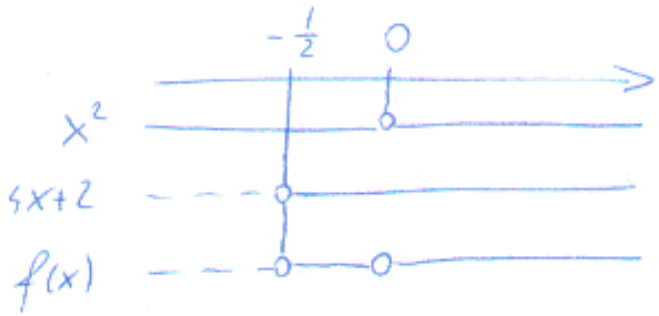
intersezioni con gli assi:

$\begin{cases} x = 0 \\ y = 0 \end{cases}$ $O(0;0)$

$\begin{cases} y = 0 \\ x^2(4x+2) = 0 \end{cases}$ $A\left(-\frac{1}{2};0\right)$

segno:

$f(x) > 0 \rightarrow x^2(4x+2) > 0$




$I.P. = \left(-\frac{1}{2};0\right) \cup (0;+\infty)$

$I.N. = \left(-\infty;-\frac{1}{2}\right)$

asintoti:

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \left(4x^3+2x^2\right) =$$

$$= \lim_{x\to\pm\infty} 4x^3 = \pm\infty$$

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{4x^3+2x^2}{x} =$$

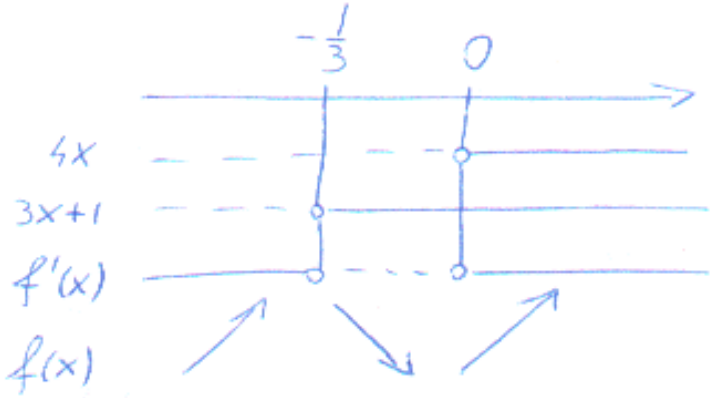
$$= \lim_{x\to\infty} 4x^2 = +\infty$$

Non esistono asintoti verticali, orizzontali, obliqui.

massimi e minimi:

$$f'(x) = 12x^2+4x = 4x(3x+1)$$

$$f'(x) > 0 \rightarrow 4x(3x+1) > 0$$



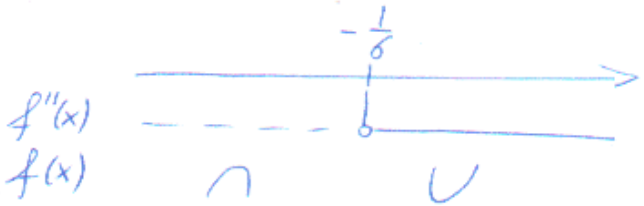

$\left(-\frac{1}{3}; \frac{2}{27}\right)$ massimo

$(0;0)$ minimo

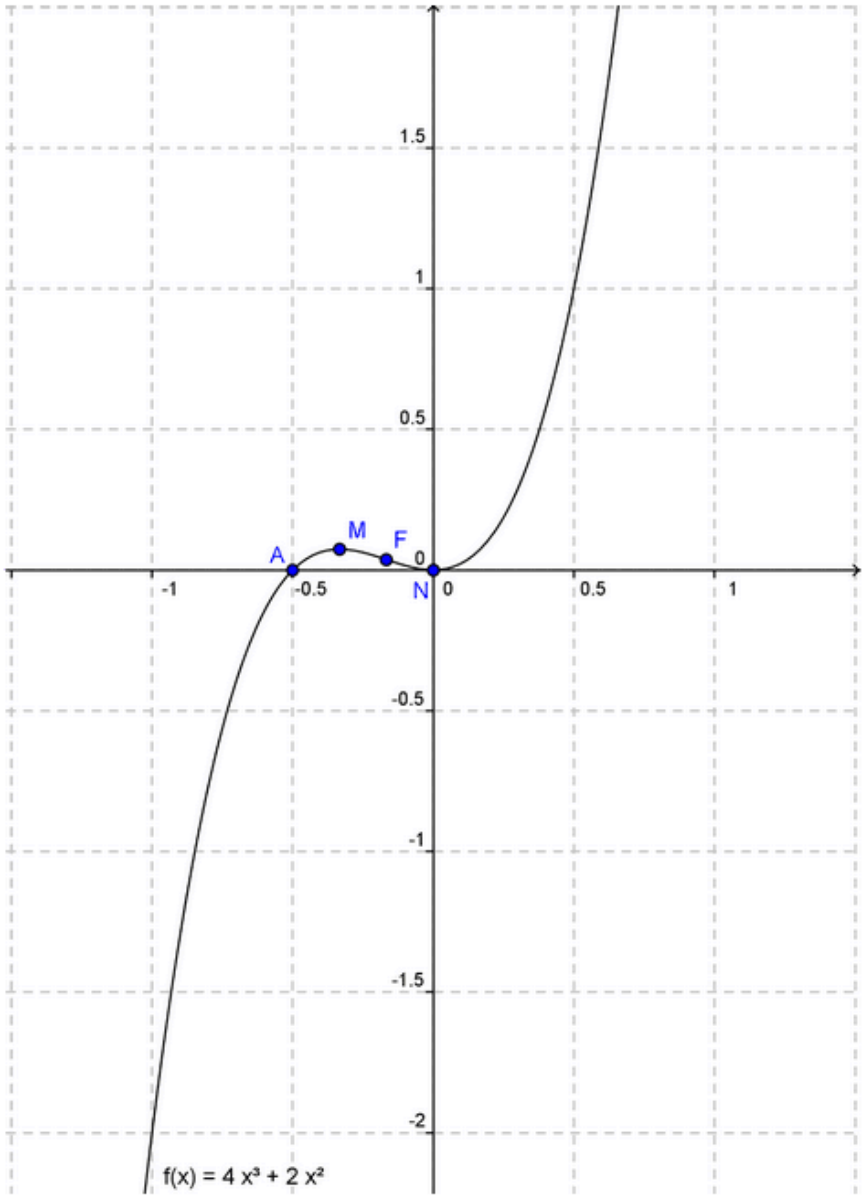
flessi:

$f''(x) = 24x + 4$

$f''(x) > 0 \rightarrow 24x + 4 > 0 \rightarrow x > -\frac{1}{6}$




$\left(-\frac{1}{6}; \frac{1}{27}\right)$ punto di flesso ascendente



$$\boxed{y = \frac{x^3 - 3x^2 + 4}{x^2}}$$

dominio:

$x \neq 0$

$D = \mathbb{R} - \{0\}$

simmetrie:

$f(-x) = -x - 3 + \frac{4}{x^2}$

$f$ né pari né dispari

intersezioni con gli assi:

$x \neq 0 \rightarrow$ non esistono intersezioni con l'asse $y$

$$\begin{cases} y = 0 \\ x^3 - 3x^2 + 4 = 0 \end{cases}$$

| | | | | |
|---|---|---|---|---|
| | 1 | -3 | 0 | 4 |
| -1 | | -1 | 4 | -4 |
| | 1 | -4 | 4 | 0 |

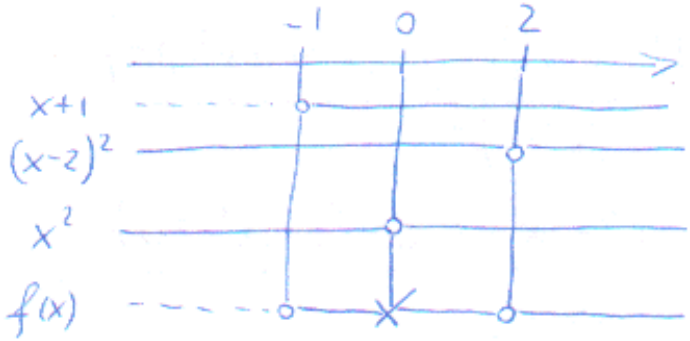
$(x+1)(x^2 - 4x + 4) = 0$

$(x+1)(x-2)^2 = 0$

$A(-1;0) \quad B(2;0)$

segno:

$$f(x)>0 \rightarrow \frac{(x+1)(x-2)^2}{x^2}>0$$




$$I.P. = (-1;0) \cup (0;2) \cup (2;+\infty)$$

$$I.N. = (-\infty;-1)$$

asintoti:

$$\lim_{x\to 0} f(x) = \frac{4}{0^+} = +\infty$$

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^3}{x^3} = 1$$

$$\lim_{x\to\infty} \left[f(x)-x\right] = \lim_{x\to\infty} \left(-3+\frac{4}{x^2}\right) = -3$$

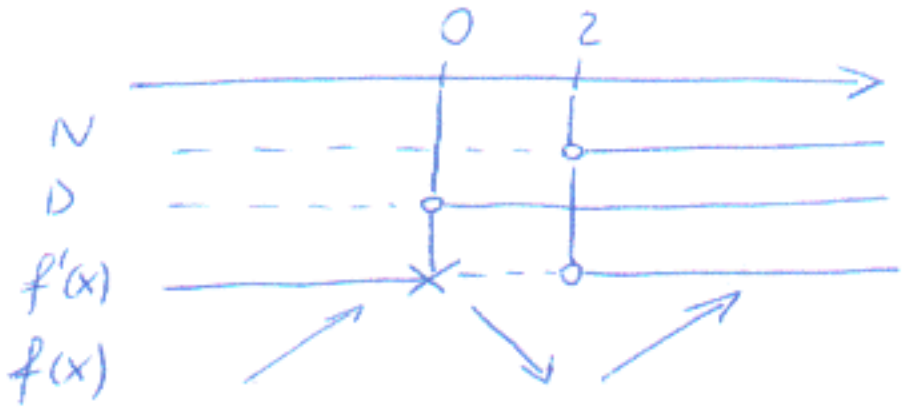
$x=0$ asintoto verticale

$y=x-3$ asintoto obliquo

massimi e minimi:

$$f'(x) = 1 - \frac{8}{x^3} = \frac{x^3-8}{x^3}$$

$$f'(x) > 0 \rightarrow \frac{(x-2)(x^2+2x+4)}{x^3} > 0$$

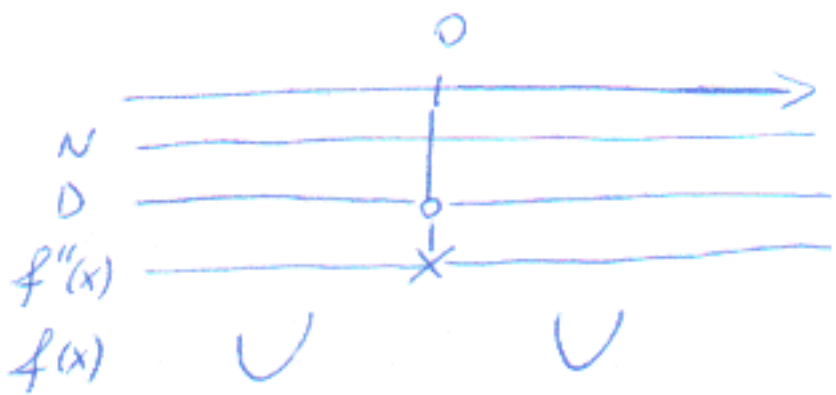



$(2;0)$ minimo

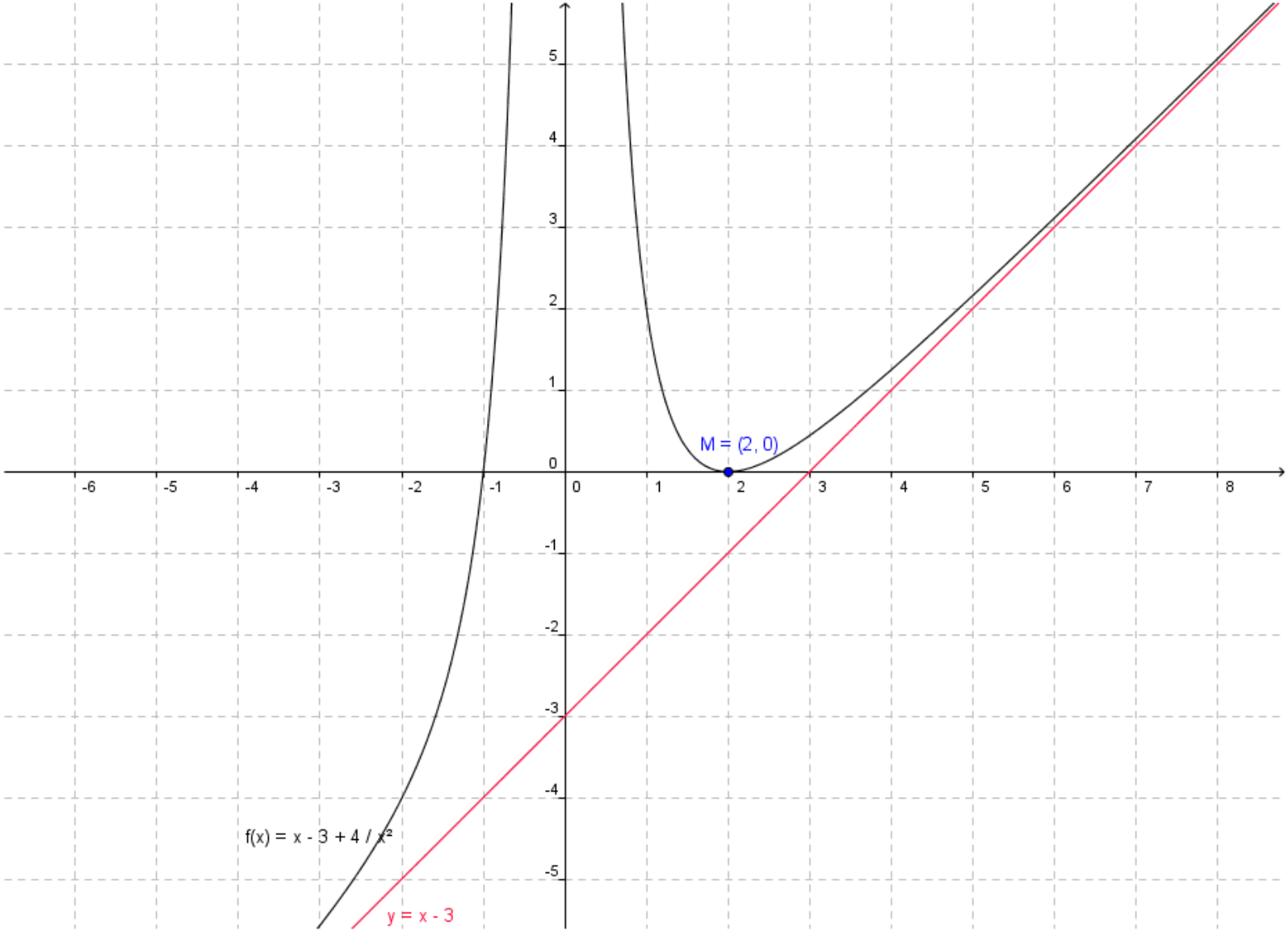
flessi :

$$f''(x) = \frac{24}{x^4}$$

$$f''(x) > 0 \rightarrow \frac{24}{x^4} > 0 \rightarrow \forall x \neq 0$$




Non ci sono punti di flesso


M = (2, 0)
f(x) = x - 3 + 4 / x²
y = x - 3

$$y = \frac{x-1}{\sqrt{x^2+1}}$$

dominio:

$x^2+1>0 \quad \forall x \in \mathbb{R}$

$D = \mathbb{R}$

simmetrie:

$f(-x) = \dfrac{-x-1}{\sqrt{x^2+1}}$

$f$ né pari né dispari

intersezioni con gli assi:

$\begin{cases} x=0 \\ y=-1 \end{cases} \quad A(0;-1)$

$\begin{cases} y=0 \\ \dfrac{x-1}{\sqrt{x^2+1}}=0 \end{cases} \quad \begin{cases} y=0 \\ x=1 \end{cases} \quad B(1;0)$

segno:

$f(x)>0 \rightarrow \dfrac{x-1}{\sqrt{x^2+1}}>0 \rightarrow x>1$

$I.P. = (1;+\infty)$

$I.N. = (-\infty;1)$

asintoti:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x-1}{|x|\sqrt{1+\frac{1}{x^2}}} =$$

$$= \lim_{x\to-\infty} \frac{\cancel{x}\left(1-\frac{1}{x}\right)}{-\cancel{x}\sqrt{1+\frac{1}{x^2}}} = -1$$

$y=-1$ asintoto orizzontale sinistro

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x-1}{|x|\sqrt{1+\frac{1}{x^2}}} =$$

$$= \lim_{x\to+\infty} \frac{\cancel{x}\left(1-\frac{1}{x}\right)}{\cancel{x}\sqrt{1+\frac{1}{x^2}}} = 1$$

$y=1$ asintoto orizzontale destro

massimi e minimi:

$$f'(x) = \frac{\sqrt{x^2+1} - \frac{x(x-1)}{\sqrt{x^2+1}}}{x^2+1}$$

$$f'(x) = \frac{\cancel{x^2}+1-\cancel{x^2}+x}{\sqrt{(x^2+1)^3}} = \frac{x+1}{\sqrt{(x^2+1)^3}}$$

$$f'(x) > 0 \quad \rightarrow \quad x > -1$$

$(-1; -\sqrt{2})$ minimo

flessi:

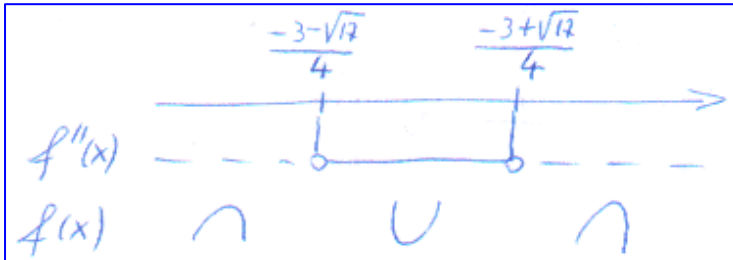
$$f''(x) = \frac{\sqrt{(x^2+1)^3} - \frac{3}{\cancel{2}} \cdot \sqrt{x^2+1} \cdot \cancel{2}x\,(x+1)}{(x^2+1)^3}$$

$$f''(x) = \frac{\sqrt{x^2+1}\,(x^2+1-3x^2-3x)}{(x^2+1)^3}$$

$$f''(x) = \frac{-2x^2-3x+1}{\sqrt{(x^2+1)^{5/2}}}$$

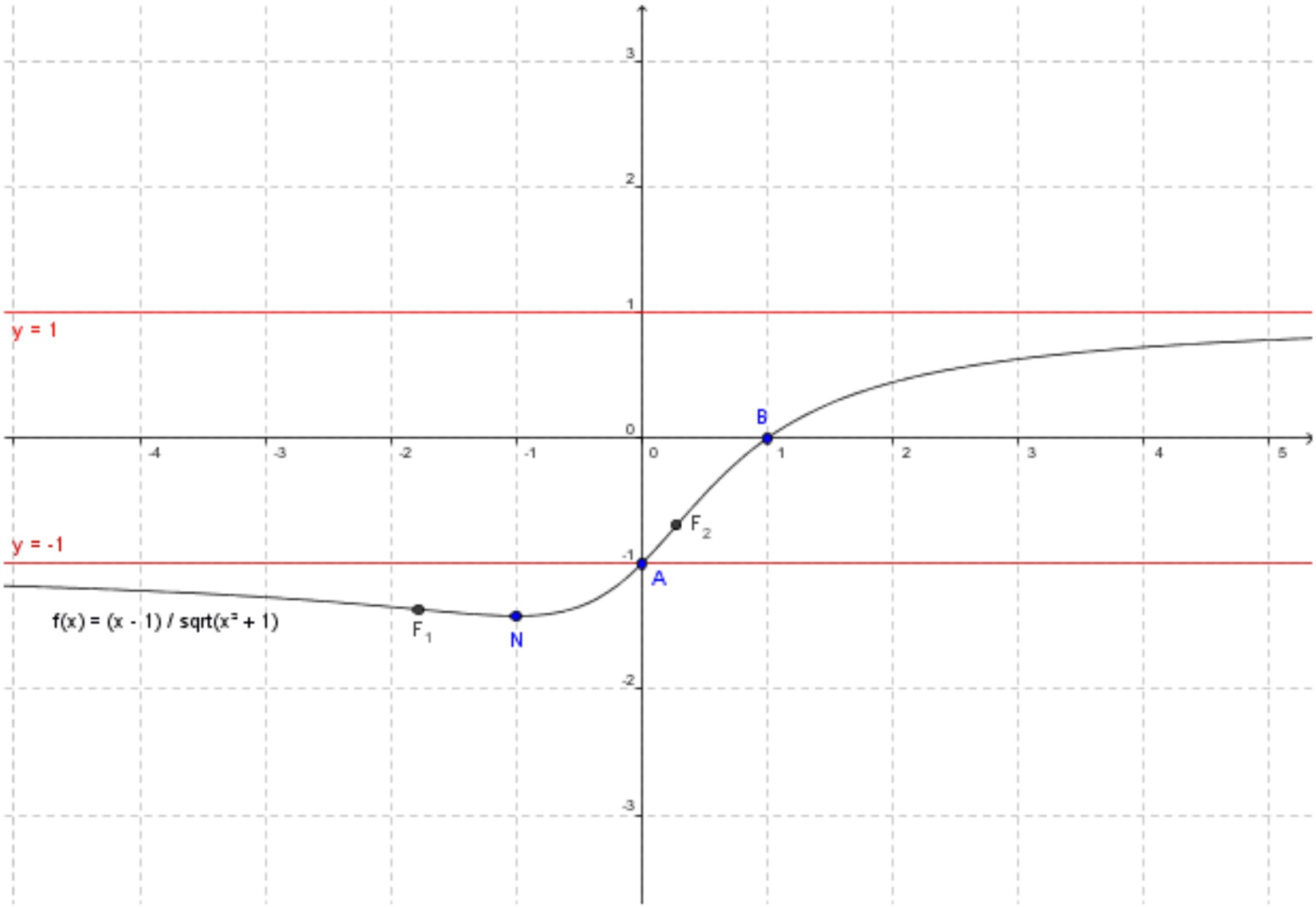
$$f''(x) > 0 \quad \rightarrow \quad -2x^2-3x+1>0$$

$$x_{1,2} = \frac{3 \pm \sqrt{17}}{-4} = \begin{cases} \frac{-3+\sqrt{17}}{4} \\ \frac{-3-\sqrt{17}}{4} \end{cases}$$




$\left(\frac{-3-\sqrt{17}}{4}; \sim -1,36\right)$ punto di flesso ascendente

$\left(\frac{-3+\sqrt{17}}{4}; \sim -0,692\right)$ punto di flesso discendente


y = 1
y = -1
f(x) = (x - 1) / sqrt(x² + 1)
A
B
N
F1
F2
-4
-3
-2
-1
0
1
2
3
4
5
3
2
1
0
-1
-2
-3

$$\boxed{y = \sqrt{1 - e^x}}$$

dominio:

$1 - e^x \geq 0$

$e^x \leq 1 \rightarrow e^x \leq e^0 \rightarrow x \leq 0$

$D = (-\infty ; 0]$

intersezioni con gli assi:

$\begin{cases} x = 0 \\ y = 0 \end{cases}$ $O(0;0)$

$\begin{cases} y = 0 \\ \sqrt{1-e^x} = 0 \end{cases} \rightarrow \begin{cases} y = 0 \\ x = 0 \end{cases}$

segno:

$f(x) > 0 \rightarrow \sqrt{1-e^x} > 0$

$I.P. = (-\infty ; 0)$

$I.N. = \emptyset$

asintoti:

$\lim_{x \to 0^-} f(x) = 0$

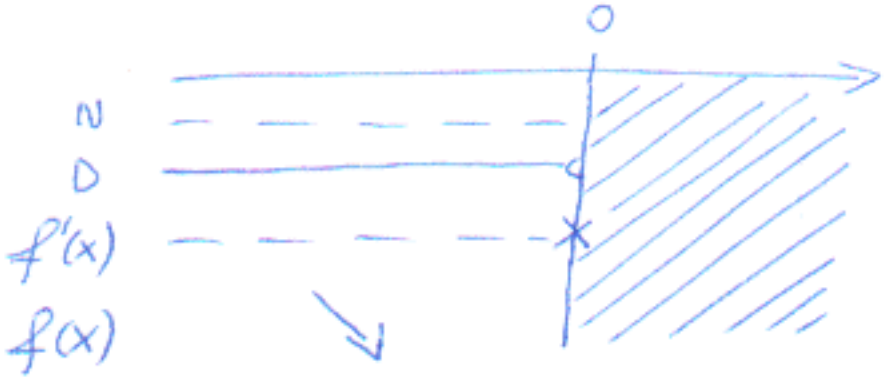
$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \sqrt{1-e^x} = 1$

$y = 1$ asintoto orizzontale sinistro

massimi e minimi:

$$f'(x) = \frac{-e^x}{2\sqrt{1-e^x}}$$

$$f'(x) > 0 \quad \rightarrow \quad \frac{-e^x}{\sqrt{1-e^x}} > 0$$



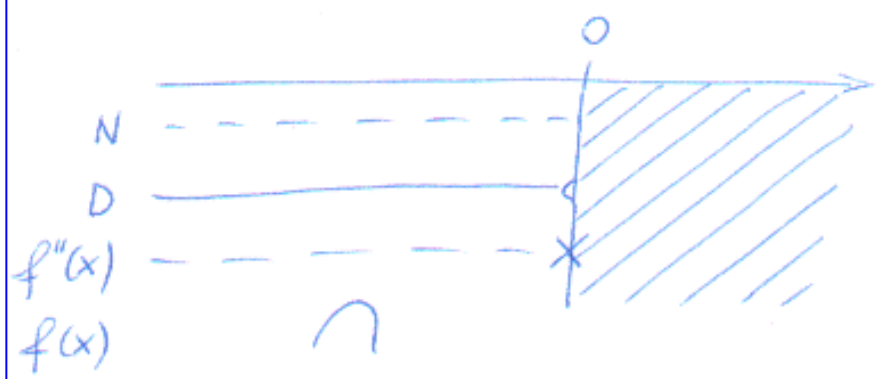

(0;0) minimo (non stazionario)

flessi:

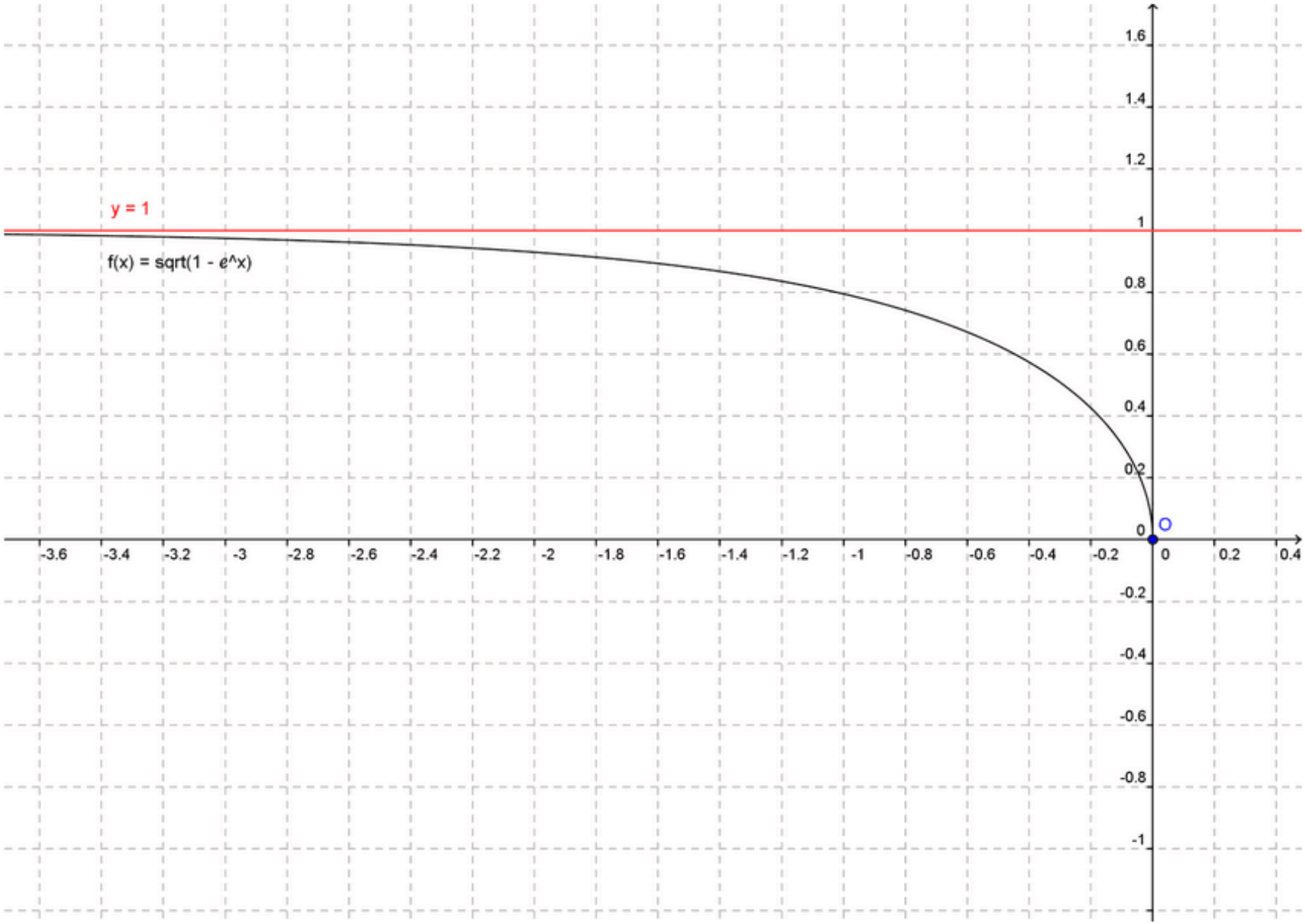
$$f''(x) = \frac{-e^x\sqrt{1-e^x} + e^x \frac{(-e^x)}{2\sqrt{1-e^x}}}{1-e^x}$$

$$f''(x) = \frac{-2e^x + 2e^{2x} - e^{2x}}{2(1-e^x)\sqrt{1-e^x}} = \frac{e^{2x} - 2e^x}{2\sqrt{(1-e^x)^3}}$$

$$f''(x) = \frac{e^x(e^x-2)}{2\sqrt{(1-e^x)^3}}$$




Non ci sono punti di flesso


y = 1
f(x) = sqrt(1 - e^x)
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-3.6
-3.4
-3.2
-3
-2.8
-2.6
-2.4
-2.2
-2
-1.8
-1.6
-1.4
-1.2
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4

$$\boxed{y = \frac{e^x}{e^x - 1}}$$

dominio:

$e^x - 1 \neq 0 \rightarrow e^x \neq 1 \rightarrow x \neq 0$

$D = \mathbb{R} - \{0\}$

simmetrie

$f(-x) = \dfrac{e^{-x}}{e^{-x} - 1}$

$f$ né pari né dispari

intersezioni con gli assi:

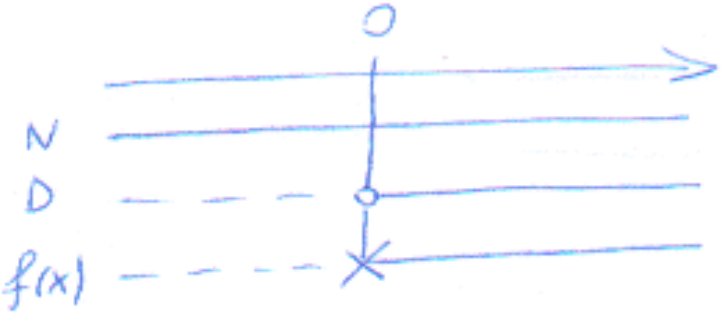
$0 \notin D$

$\begin{cases} y = 0 \\ \dfrac{e^x}{e^x - 1} = 0 \end{cases} \rightarrow \emptyset$

Non ci sono intersezioni con gli assi

segno:

$$f(x) > 0 \rightarrow \frac{e^x}{e^x - 1} > 0$$




$$I.P. = (0; +\infty)$$

$$I.N. = (-\infty; 0)$$

asintoti:

$$\lim_{x \to 0^-} \frac{e^x}{e^x - 1} = \frac{1}{0^-} = -\infty$$

$$\lim_{x \to 0^+} \frac{e^x}{e^x - 1} = \frac{1}{0^+} = +\infty$$

$x = 0$ asintoto verticale

$$\lim_{x \to +\infty} \frac{\cancel{e^x}}{\cancel{e^x}\left(1 - \frac{1}{e^x}\right)} = 1$$

$y = 1$ asintoto orizzontale destro

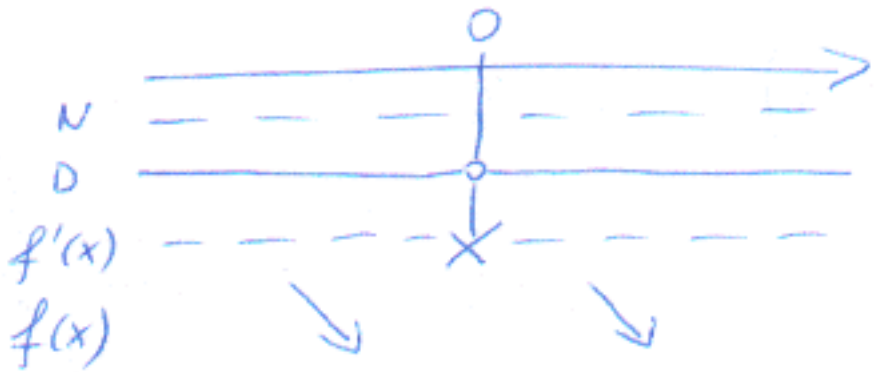
$$\lim_{x \to -\infty} \frac{e^x}{e^x - 1} = 0$$

$y = 0$ asintoto orizzontale sinistro

massimi e minimi:

$$f'(x) = \frac{e^{2x} - e^{x} - e^{2x}}{(e^{x}-1)^2} = \frac{-e^{x}}{(e^{x}-1)^2}$$

$$f'(x) > 0 \quad \rightarrow \quad \frac{-e^{x}}{(e^{x}-1)^2} > 0$$



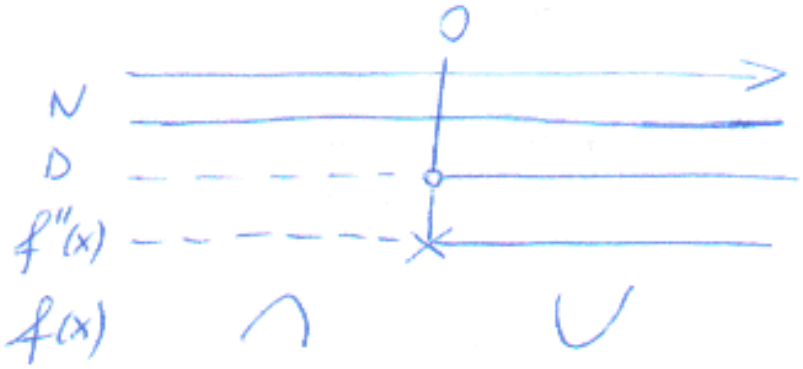

Non ci sono massimi e minimi

flessi :

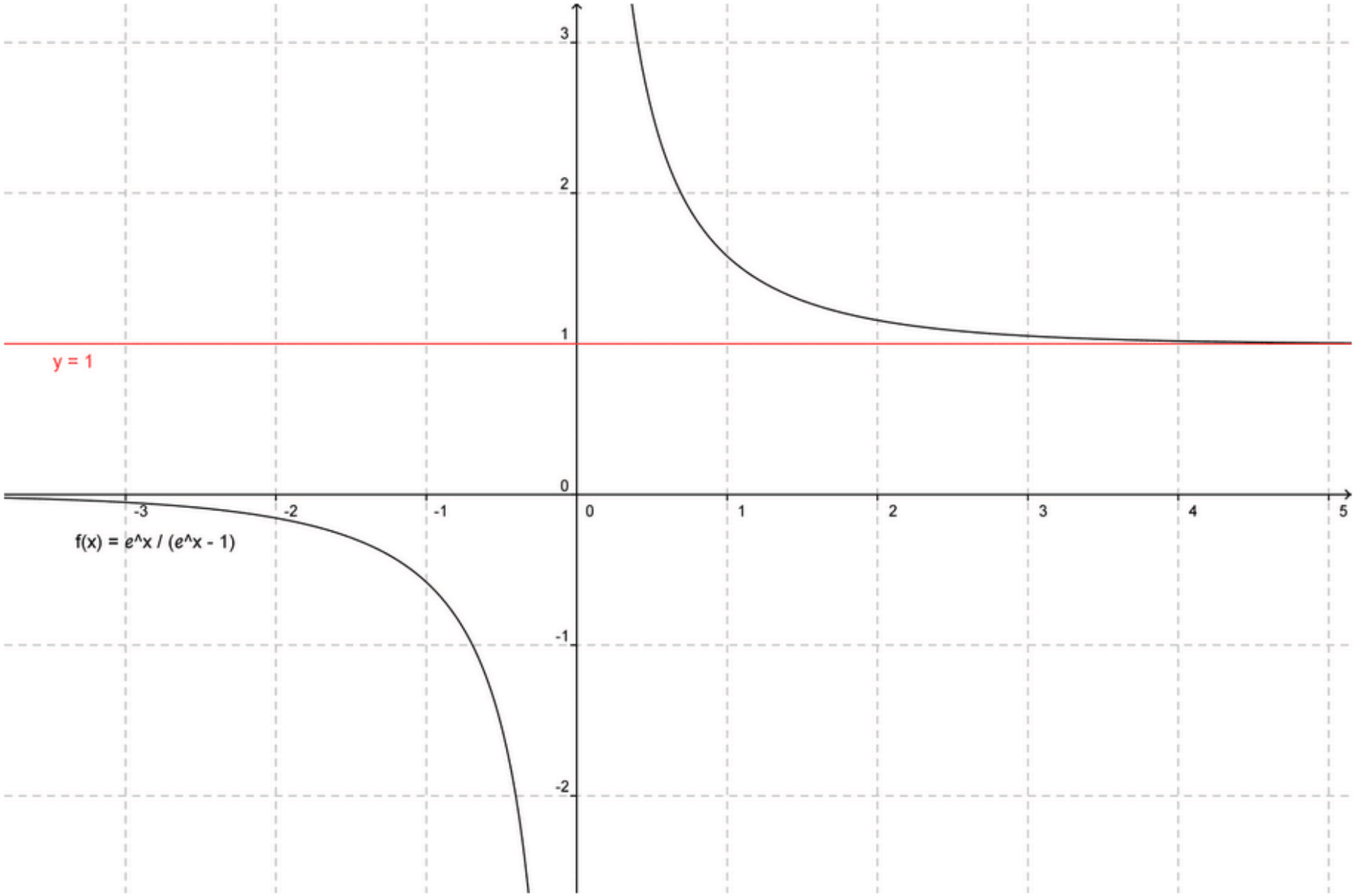
$$f''(x) = \frac{-e^x(e^x-1)^2 + 2e^x(e^x-1)\cdot e^x}{(e^x-1)^4}$$

$$f''(x) = \frac{-e^{2x} + e^x + 2e^{2x}}{(e^x-1)^3} = \frac{e^{2x} + e^x}{(e^x-1)^3}$$

$$f''(x) > 0 \rightarrow \frac{e^x(e^x+1)}{(e^x-1)^3} > 0$$




Non ci sono flessi


y = 1
f(x) = e^x / (e^x - 1)
3
2
1
0
-1
-2
-3
-2
-1
0
1
2
3
4
5

$$y = \frac{1}{x}\log^2 x$$

dominio:

$x > 0$

$D = (0; +\infty)$

intersezioni con gli assi:

$0 \notin D$

$$\begin{cases} y = 0 \\ \frac{1}{x}\log^2 x = 0 \end{cases} \rightarrow \begin{cases} y = 0 \\ x = 1 \end{cases}$$

$A(1;0)$

segno:

$$f(x) > 0 \rightarrow \frac{\log^2 x}{x} > 0$$

$I.P. = (0; +1) \cup (+1; +\infty)$

$I.N. = \emptyset$

asintoti:

$$\lim_{x \to 0^+} \frac{\log^2 x}{x} = +\infty$$

$$\lim_{x \to +\infty} \frac{\log^2 x}{x} = \lim_{x \to +\infty} \frac{\frac{1}{x} \cdot 2\log x}{1} =$$

$$= \lim_{x \to +\infty} \frac{2}{x} = 0$$

$x = 0$ asintoto verticale

$y = 0$ asintoto orizzontale

massimi e minimi:

$$f'(x) = \frac{2\log x \cdot \frac{1}{\cancel{x}} \cdot \cancel{x} - \log^2 x}{x^2}$$

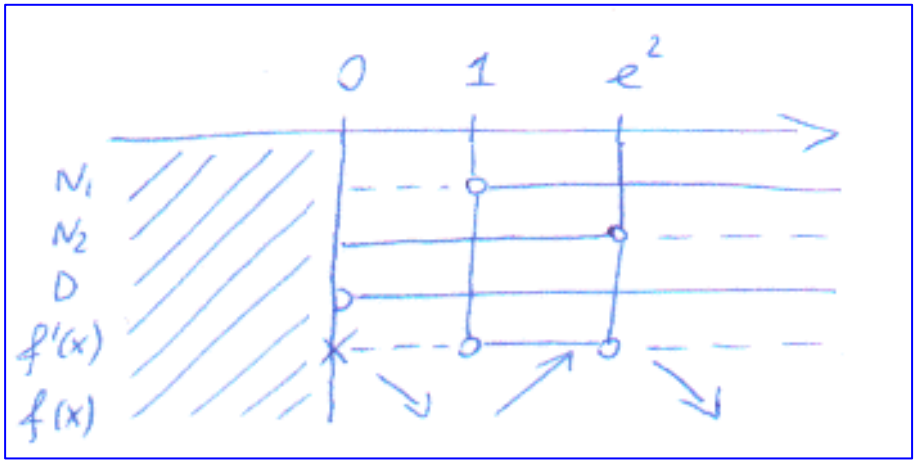
$$f'(x) = \frac{\log x\,(2 - \log x)}{x^2}$$

$f'(x) > 0$

$N_1 > 0 \rightarrow \log x > 0 \rightarrow x > 1$

$N_2 > 0 \rightarrow 2 - \log x > 0 \rightarrow \log x < \log e^2 \rightarrow x < e^2$

$D > 0 \rightarrow x^2 > 0 \qquad \forall x \in (0; +\infty)$




$(1; 0)$ minimo

$\left(e^2; \frac{4}{e^2}\right)$ massimo

flessi:

$$f''(x) = \frac{\left(\frac{2}{x} - \frac{2}{x}\log x\right)x^2 - 2x\log x\,(2-\log x)}{x^4}$$

$$f''(x) = \frac{2x - 2x\log x - 4x\log x + 2x\log^2 x}{x^4}$$

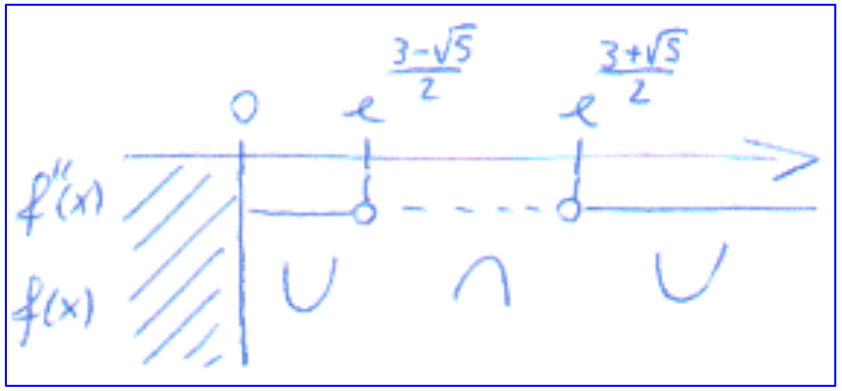
$$f''(x) = \frac{2\cancel{x}(\log^2 x - 3\log x + 1)}{x^{\cancel{4}\,3}}$$

$$f''(x) > 0 \rightarrow \log^2 x - 3\log x + 1 > 0$$

$$\log x = \frac{3 \pm \sqrt{5}}{2}$$

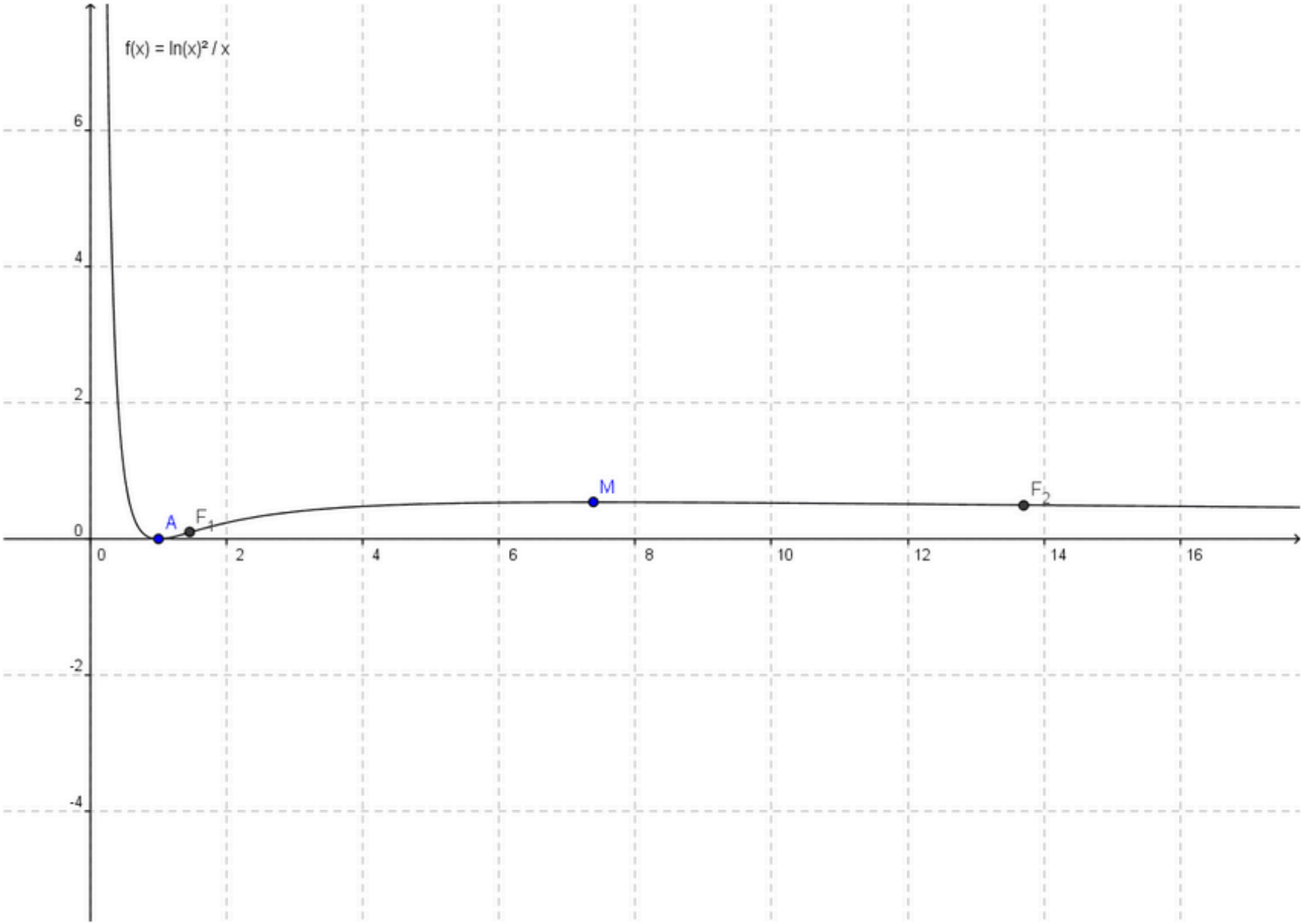
$$\log x < \frac{3-\sqrt{5}}{2} \quad \vee \quad \log x > \frac{3+\sqrt{5}}{2}$$

$$0 < x < e^{\frac{3-\sqrt{5}}{2}} \quad \vee \quad x > e^{\frac{3+\sqrt{5}}{2}}$$




$\left(e^{\frac{3-\sqrt{5}}{2}};\ \sim 0{,}0996\right)$ punto di flesso discendente

$\left(e^{\frac{3+\sqrt{5}}{2}};\ \sim 0{,}49998\right)$ punto di flesso ascendente


f(x) = ln(x)² / x
6
4
2
0
-2
-4
0
2
4
6
8
10
12
14
16
A
F1
M
F2

$$f(x) = \frac{x^2}{x^2 - |x-2|}$$

$$|x-2| \begin{cases} x-2 & \text{se } x-2 \geq 0 \rightarrow x \geq 2 \\ 2-x & \text{se } x-2 < 0 \rightarrow x < 2 \end{cases}$$

$$\begin{cases} f(x) = \dfrac{x^2}{x^2 - x + 2} \\ x \geq 2 \end{cases} \qquad \begin{cases} f(x) = \dfrac{x^2}{x^2 + x - 2} \\ x < 2 \end{cases}$$

1) DOMINIO:

$x^2 - x + 2 \neq 0 \ (\Delta < 0)$

$D_1 = [2; +\infty)$

$x^2 + x - 2 \neq 0 \begin{cases} x \neq 1 \\ x \neq -2 \end{cases}$

$D_2 = \{x : x < 2, x \neq 1 \vee x \neq -2\}$

2) SIMMETRIE: NO!

3) INT. CON GLI ASSI:

$f(x) = 0 \rightarrow x = 0 \ (< 2)$ No

$f(x) = 0 \rightarrow x = 0 \rightarrow (0,0) \in f(x)$

4) SEGNO:

$f(x) \geq 0$

$\left.\begin{array}{l} N \geq 0 \ \forall x \geq 2 \\ D > 0 \ \forall x \geq 2 \end{array}\right\} \dfrac{f(x) > 0}{\forall x \geq 2}$

$f(x) \geq 0$

$N \geq 0 \ \forall x < 2$

$D > 0 \rightarrow x < -2 \vee x > 1$

| | | $-2$ | | $0$ | | $1$ | | $2$ | $x$ |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | $+$ | | $-$ | | $-$ | | $+$ | | $+$ |

5) LIMITI:

$x \geq 2$

$$\lim_{x \to +\infty} f(x) = 1$$

$y = 1$ ASINTOTO ORIZZONT.

$x < 2$

$$\lim_{x \to -\infty} f(x) = 1$$

$y = 1$ ASINTOTO ORIZZONTALE

$$\lim_{x \to 1^{\pm}} f(x) = \lim_{x \to -2^{\pm}} f(x) = \infty$$

$x = 1$, $x = -2$ ASINTOTI VERTICALI

6) DERIVATE:

$$f'(x) = \frac{2x(x^2-x+2)-x^2(2x-1)}{(x^2-x+2)^2}$$

$f'(x) \geq 0$

$2x^3-2x^2+4x-2x^3+x^2 \geq 0$

$-x^2+4x \geq 0$

$\begin{cases} x(-x+4) \geq 0 \\ x \geq 2 \end{cases}$ $\quad 2 \leq x \leq 4$

$$f'(x) = \frac{2x(x^2+x-2)-x^2(2x+1)}{(x^2+x-2)^2}$$

$f'(x) \geq 0$

$2x^3+2x^2-4x-2x^3-x^2 \geq 0$

$x^2-4x \geq 0$

$\begin{cases} x(x-4) \geq 0 \\ x < 2 \end{cases}$

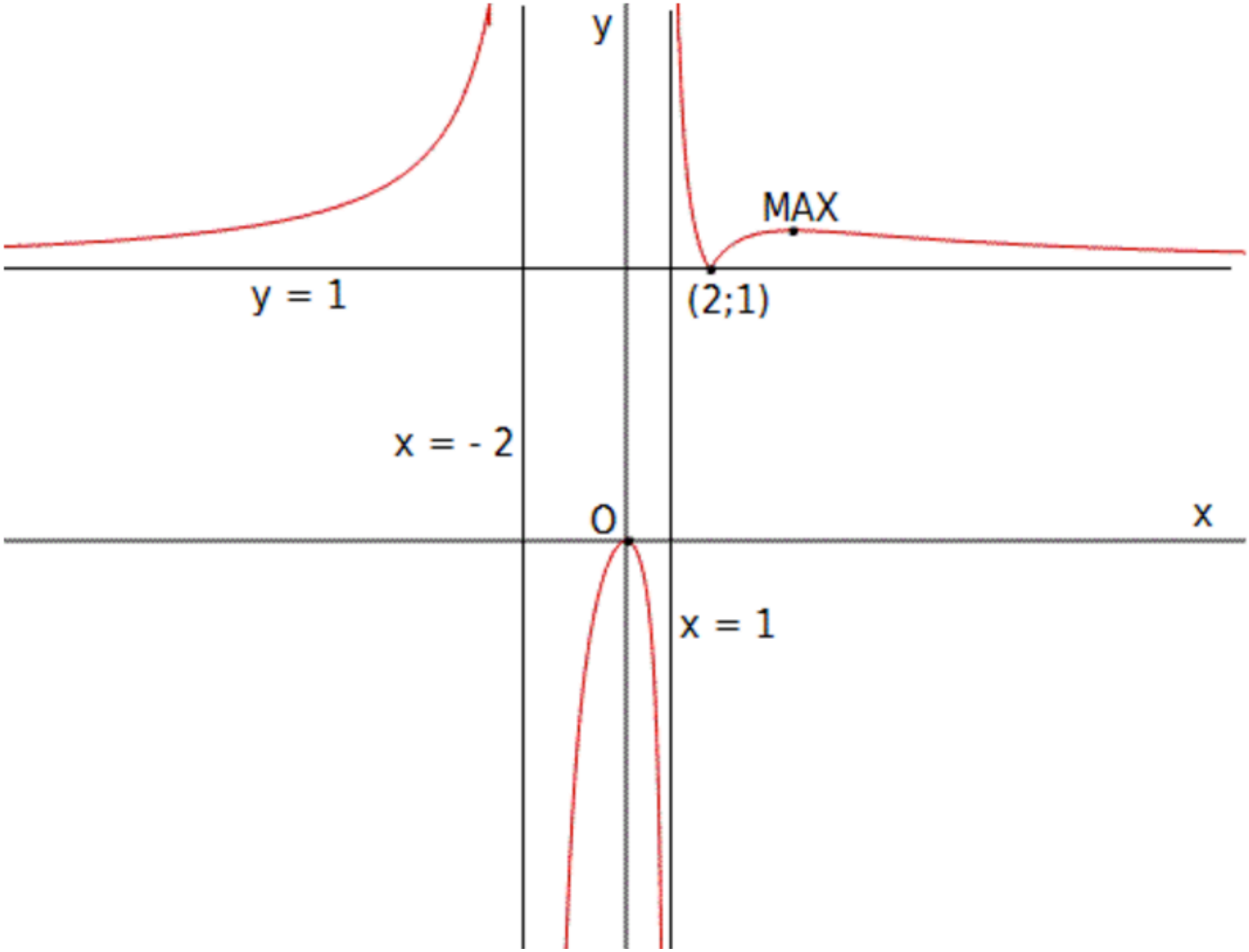
$x \leq 0$

| | | 0 | | 2 | | 4 | |
|---|---|---|---|---|---|---|---|
| $f'(x)$ | + | | − | | + | | − |
| $f(x)$ | ↗ | MAX | ↘ | P | ↗ | MAX | ↘ |

$P(2;1)$

PUNTO ANGOLOSO

MAX in $0$ e $(4;\frac{8}{7})$


y
MAX
y = 1
(2;1)
x = - 2
O
x
x = 1

$$\boxed{y = e^{\left|\frac{1}{1-x}-1\right|}}$$

Iniziamo con il dominio:

Abbiamo un denominatore, dobbiamo richiedere che esso sia diverso da zero:

$$1 - x \neq 0 \iff x \neq 1$$

Il dominio è quindi:

$$\mathrm{dom}(f) = (-\infty, 1) \cup (1, +\infty)$$

## Segno della funzione

La funzione è sempre positiva nel dominio, lo possiamo asserire perché la funzione è esponenziale.

## Limiti agli estremi del dominio

$$\lim_{x \to -\infty} e^{\left|\frac{1}{1-x}-1\right|} = e$$

Abbiamo un asintoto orizzontale sinistro di equazione $y = e$

$$\lim_{x \to +\infty} e^{\left|\frac{1}{1-x}-1\right|} = e$$

$y = e$ è anche asintoto orizzontale sinistro. Non abbiamo asintoti obliqui.

$$\lim_{x \to 1^-} e^{\left|\frac{1}{1-x}-1\right|} = +\infty$$

$$\lim_{x \to 1^+} e^{\left|\frac{1}{1-x}-1\right|} = +\infty$$

$x = 1$ è un asintoto verticale.

Per studiare la monotonia della funzione, dobbiamo calcolare prima la derivata prima. Ci conviene scrivere la funzione in modo diverso, utilizzando la definizione di valore assoluto. Cominciamo con lo studio del segno dell'argomento del valore assoluto.

$$\frac{1}{1-x} - 1 > 0 \iff \frac{1-1+x}{1-x} > 0 \iff \frac{x}{1-x} > 0$$

Studiamo separatamente il segno del numeratore e del denominatore:

$$x > 0$$

$$1 - x > 0 \iff x < 1$$

Tabuliamo i segno:

- - - - - - - - - 0 + + + + + + + + +
+ + + + + + + + + 0 + + + +1 - - - -
- - - - - - - - - 0 + + + +1 - - - -

di conseguenza la funzione è:

$$f(x) = \begin{cases} e^{\frac{x}{1-x}} & \text{se } 0 < x < 1 \\ e^{-\frac{x}{1-x}} & \text{se } x < 0 \vee x > 1 \end{cases}$$

Possiamo calcolare la derivata prima:

$$f'(x) = \begin{cases} \frac{e^{\frac{x}{1-x}}}{(x-1)^2} & \text{se } 0 < x < 1 \\ -\frac{e^{\frac{x}{1-x}}}{(x-1)^2} & \text{se } x < 0 \vee x > 1 \end{cases}$$

Abbiamo dei dubbi di derivabilità nel punto 0:

$$f'_-(0) = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} =$$

$$f'_-(0) = \lim_{h \to 0^-} \frac{e^{-\frac{h}{1-h}} - 1}{h}$$

Moltiplichiamo e dividiamo per $-(1-h)$ il denominatore:

$$f'_-(0) = \lim_{h \to 0^-} \frac{e^{-\frac{h}{1-h}} - 1}{-(1-h)\frac{h}{-(1-h)}} =$$

$$f'_-(0) = \lim_{h \to 0^-} -\frac{1}{1-h} \cdot \lim_{h \to 0^-} \frac{e^{\frac{h}{1-h}} - 1}{\frac{h}{-(1-h)}} = -1$$

mentre:

$$f'_+(0) = \lim_{h \to 0^+} \frac{e^{\frac{h}{1-h}} - 1}{h}$$

moltiplichiamo e dividiamo per $1 - h$ il denominatore:

$$f'_+(0) = \lim_{h \to 0^+} \frac{e^{\frac{h}{1-h}} - 1}{(1-h)\frac{h}{(1-h)}} = 1$$

Il limite destro e il limite sinistro sono finiti ma non coincidono, quindi $x = 0$ è un punto di non derivabilità. Abbiamo un punto angoloso.

Lo studio della derivata prima per i massimi e i minimi è immediato. Nota infatti che la derivata prima è positiva per $0 < x < 1$ mentre è negativa per x<0 o x>1, quindi la funzione è:

decrescente in $x < 0 \vee x > 1$
crescente in $0 < x < 1$

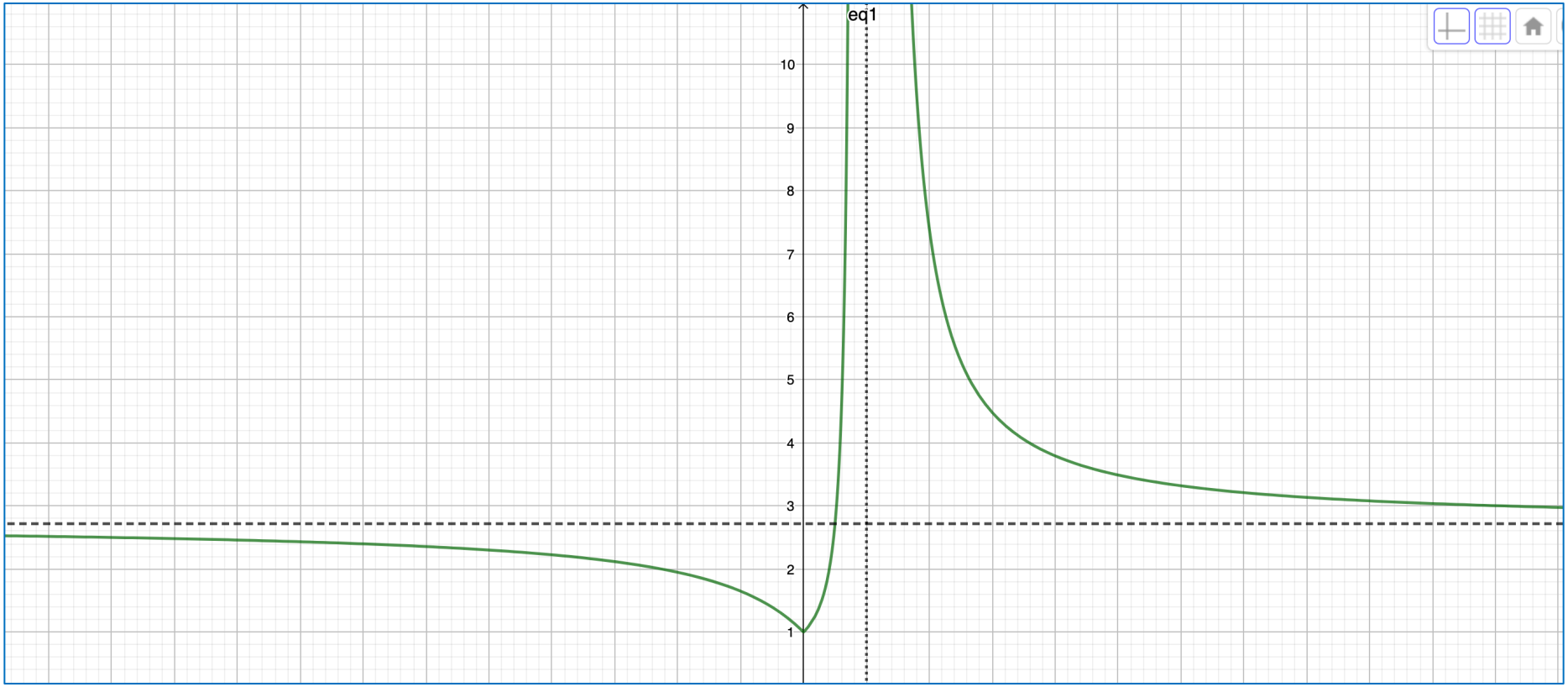
Per la convessità dobbiamo studiare la derivata seconda 🙂

$$f''(x) = \begin{cases} \frac{e^{\frac{x}{1-x}}(3-2x)}{(x-1)^4} & \text{se } 0 < x < 1 \\ \\ \frac{e^{\frac{x}{1-x}}(2x-1)}{(x-1)^4} & \text{se } x < 0 \vee x > 1 \end{cases}$$

Studiare il segno della derivata seconda è sempre tranquilla, questo perché per $0 < x < 1$ essa è positiva, quindi la funzione è convessa.

Per $x < 0$ la derivata seconda è negativa, quindi la funzione è concava.

Per $x > 1$ la derivata seconda è positiva, quindi la funzione è convessa


eq1
10
9
8
7
6
5
4
3
2
1

$$y = \log\left|x^2 - 5x + 4\right|$$

dominio:

$$x^2 - 5x + 4 \neq 0$$

$$(x-4)(x-1) \neq 0$$

$$D = \mathbb{R} - \{1; 4\}$$

intersezioni con gli assi:

$$\begin{cases} x = 0 \\ y = \log 4 \end{cases} \qquad A(0; \log 4)$$

$$\begin{cases} y = 0 \\ \log|x^2 - 5x + 4| = 0 \end{cases}$$

$$|x^2 - 5x + 4| = 1$$

$x^2 - 5x + 4 = 1$ ✓ $\qquad$ $x^2 - 5x + 4 = -1$

$x^2 - 5x + 3 = 0$ ✓ $\qquad$ $x^2 - 5x + 5 = 0$

$x_{1,2} = \frac{5 \pm \sqrt{13}}{2} \begin{cases} \sim 0{,}697 \\ \sim 4{,}30 \end{cases}$ ✓ $\qquad$ $x_{1,2} = \frac{5 \pm \sqrt{5}}{2} \begin{cases} \sim 1{,}38 \\ \sim 3{,}62 \end{cases}$

$B\left(\frac{5 - \sqrt{13}}{2}; 0\right)$ $\qquad$ $E\left(\frac{5 - \sqrt{5}}{2}; 0\right)$

$C\left(\frac{5 + \sqrt{13}}{2}; 0\right)$ $\qquad$ $F\left(\frac{5 + \sqrt{5}}{2}; 0\right)$

segno:

$f(x) > 0$

$\log|x^2-5x+4| > \log 1$

$|x^2-5x+4| > 1$

$x^2-5x+4 < -1 \quad \vee \quad x^2-5x+4 > 1$

$x^2-5x+5 < 0 \quad \vee \quad x^2-5x+3 > 0$

$\frac{5-\sqrt{5}}{2} < x < \frac{5+\sqrt{5}}{2} \quad \vee \quad x < \frac{5-\sqrt{13}}{2} \quad \vee \quad x > \frac{5+\sqrt{13}}{2}$

$I.P. = \left(-\infty; \frac{5-\sqrt{13}}{2}\right) \cup \left(\frac{5-\sqrt{5}}{2}; \frac{5+\sqrt{5}}{2}\right) \cup \left(\frac{5+\sqrt{13}}{2}; +\infty\right)$

$I.N. = \left(\frac{5-\sqrt{13}}{2}; 1\right) \cup \left(1; \frac{5-\sqrt{5}}{2}\right) \cup \left(\frac{5+\sqrt{5}}{2}; 4\right) \cup \left(4; \frac{5+\sqrt{13}}{2}\right)$

asintoti:

$\lim_{x \to 1} f(x) = -\infty$ $\quad x=1$ asintoto verticale

$\lim_{x \to 4} f(x) = -\infty$ $\quad x=4$ asintoto verticale

$\lim_{x \to \infty} f(x) = +\infty$

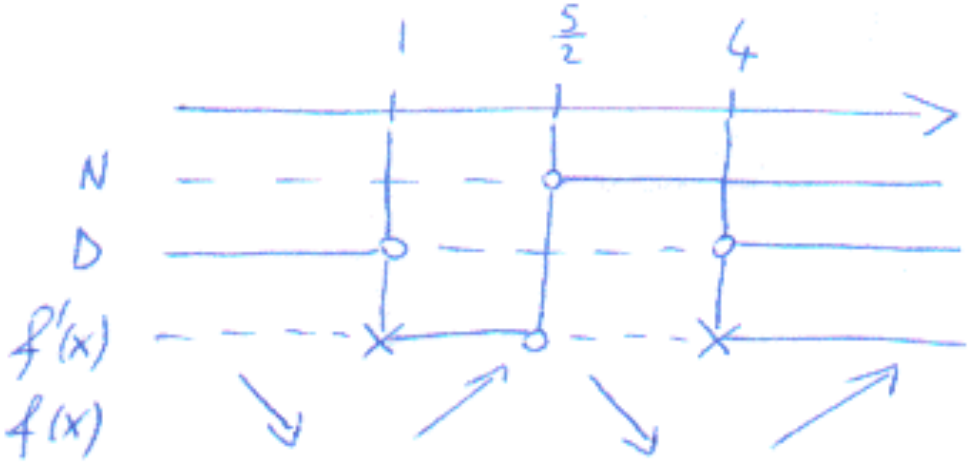
$$\lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{2x-5}{x^2-5x+4} = 0$$

Non ci sono asintoti orizzontali o obliqui.

massimi e minimi :

$$f'(x) = \frac{2x-5}{x^2-5x+4}$$

$$f'(x) > 0 \rightarrow \frac{2x-5}{(x-4)(x-1)} > 0$$



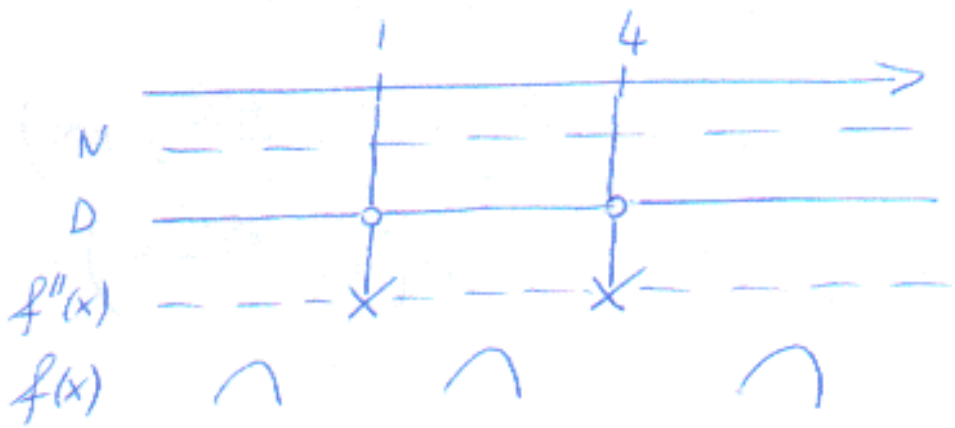

$$\left(\frac{5}{2}; \log\frac{9}{4}\right)$$ massimo

Flessi:

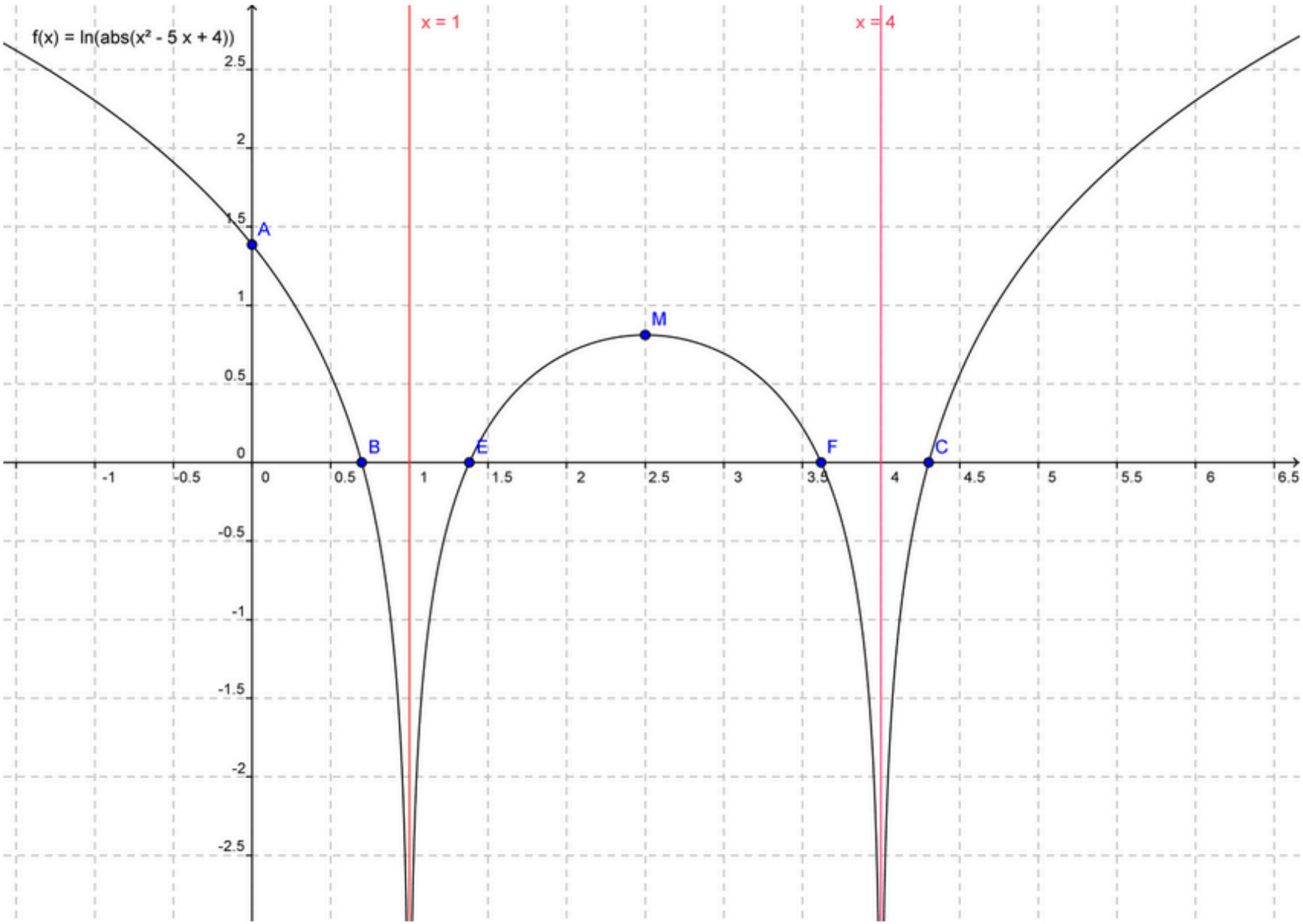
$$f''(x) = \frac{2x^2 - \cancel{10x} + 8 - 4x^2 + \cancel{10x} + 10x - 25}{(x^2 - 5x + 4)^2}$$

$$f''(x) = \frac{-2x^2 + 10x - 17}{(x^2 - 5x + 4)^2}$$

$$f''(x) > 0 \rightarrow \frac{-(2x^2 - 10x + 17)}{(x-4)^2(x-1)^2} > 0$$




Non ci sono flessi


f(x) = ln(abs(x² - 5 x + 4))
x = 1
x = 4
A
B
E
M
F
C
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
6.5

$$\boxed{y = 2senx - sen2x}$$

dominio:

$D = \mathbb{R}$

simmetrie:

$f(-x) = -2\sin x + \sin 2x = -f(x)$

$f$ è dispari

periodicità:

$y_1 = 2\sin x$ ha periodicità $2\pi$

$y_2 = -\sin 2x$ ha periodicità $\pi$

$y = 2\sin x - \sin 2x$ ha periodicità $2\pi$

intersezioni con gli assi:

$$\begin{cases} x=0 \\ y=0 \end{cases} \qquad O(0;0)$$

$$\begin{cases} y=0 \\ 2\sin x - \sin 2x = 0 \end{cases}$$

$$2\sin x - 2\sin x \cos x = 0$$

$$2\sin x\,(1-\cos x) = 0$$

$$\sin x = 0 \rightarrow x = k\pi,\ k \in \mathbb{Z}$$

$$\cos x = 1 \rightarrow x = 2k\pi$$

$A(\pi;0)$ $B(2\pi;0)$ $C(3\pi;0)$ etc.

segno:

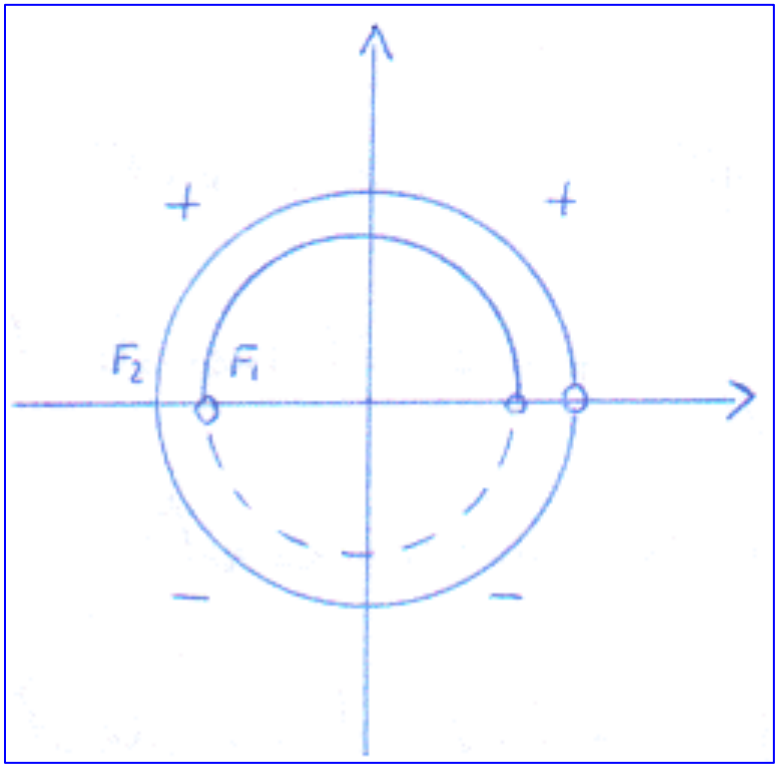
$f(x)>0 \rightarrow 2\sin x\,(1-\cos x)>0$

$F_1>0 \rightarrow 2\sin x>0$

$2k\pi<x<\pi+2k\pi$

$F_2>0 \rightarrow 1-\cos x>0 \rightarrow \cos x<1$

$x\neq 2k\pi$




$$I.P. = \{x\in\mathbb{R} \,/\, 2k\pi<x<\pi+2k\pi,\ k\in\mathbb{Z}\}$$

$$I.N. = \{x\in\mathbb{R} \,/\, \pi+2k\pi<x<2\pi+2k\pi,\ k\in\mathbb{Z}\}$$

asintoti:

Non ci sono asintoti

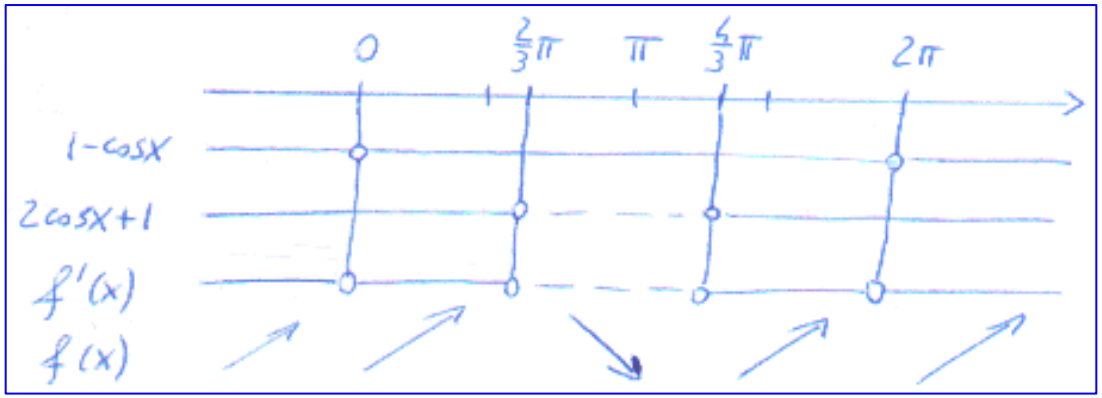
massimi e minimi:

$$f'(x) = 2\cos x - 2\cos 2x = 2(\cos x - \cos 2x)$$

$$f'(x) = 2(\cos x - 2\cos^2 x + 1)$$

$$f'(x) = 2(1-\cos x)(2\cos x + 1)$$

$$f'(x) > 0$$




$\left(\frac{2}{3}\pi + 2k\pi ; \frac{3\sqrt{3}}{2}\right)$ massimi

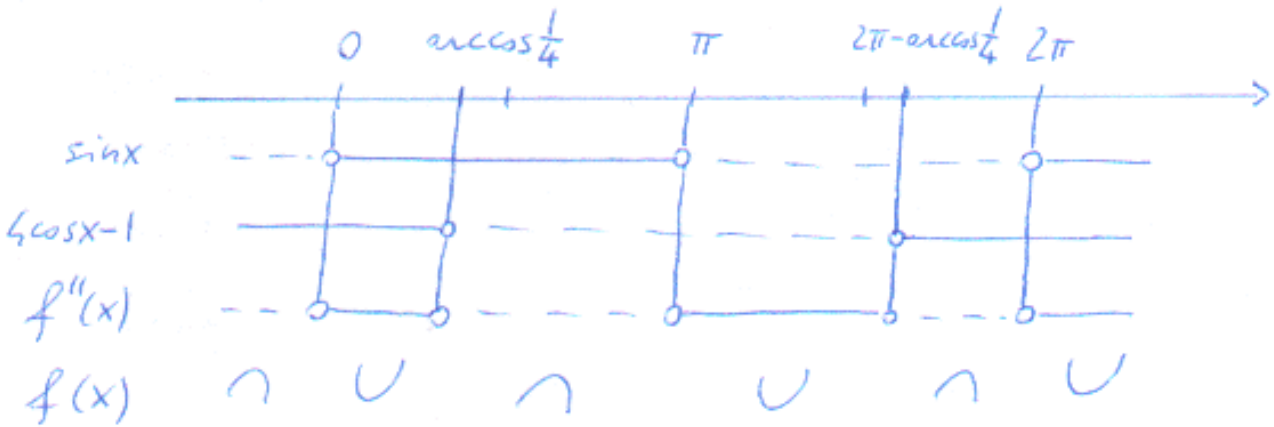
$\left(\frac{4}{3}\pi + 2k\pi ; -\frac{3\sqrt{3}}{2}\right)$ minimi

$(2k\pi ; 0)$ punti di flesso a tangente orizzontale

flessi:

$$f''(x) = -2\sin x + 4\sin 2x = 8\sin x\cos x - 2\sin x$$

$$f''(x) = 2\sin x\,(4\cos x - 1)$$

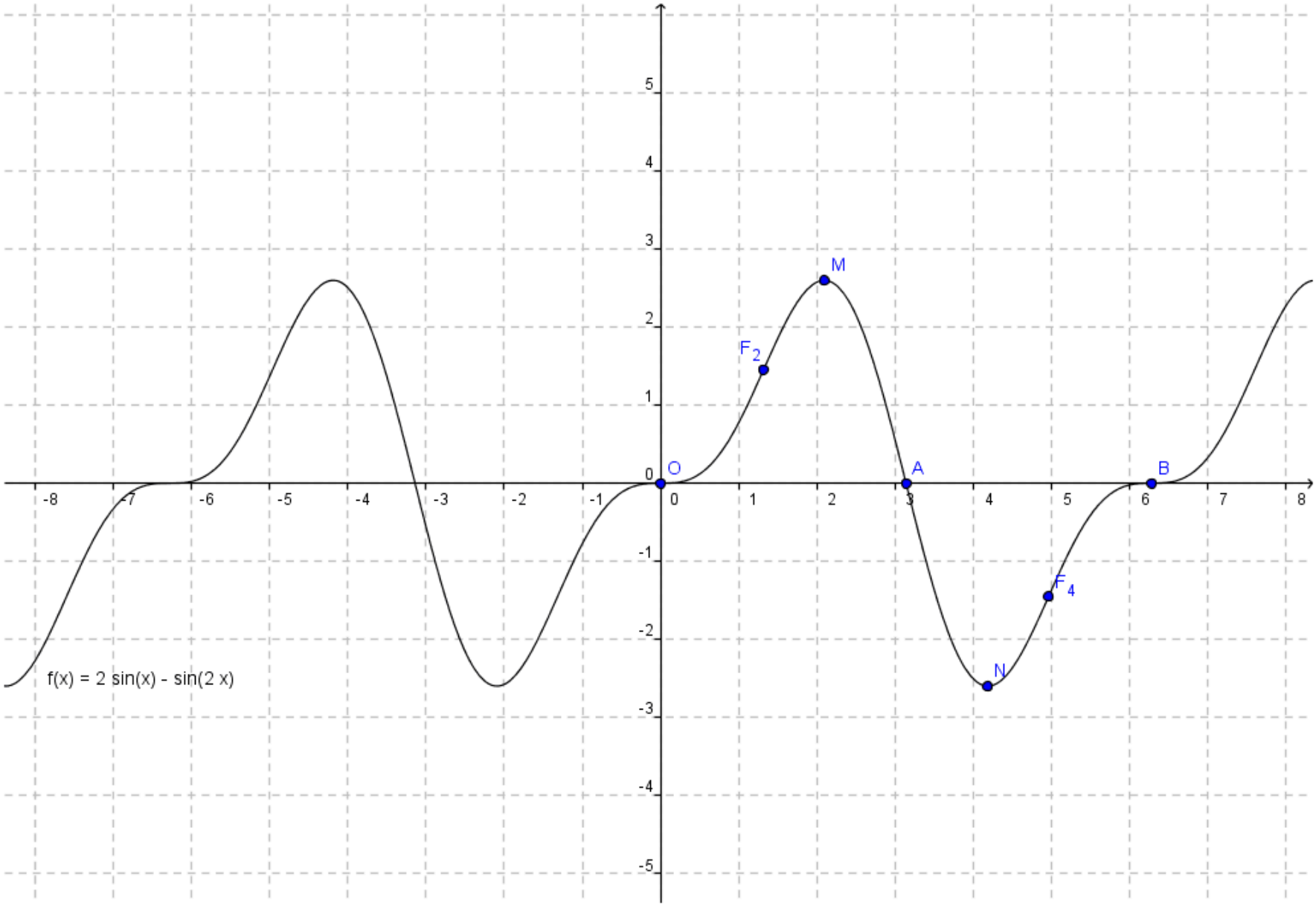



$(2k\pi\,;\,0)$ flessi ascendenti

$\left(\arccos\frac{1}{4} + 2k\pi\,;\,\frac{3\sqrt{15}}{8}\right)$ flessi discendenti

$(\pi + 2k\pi\,;\,0)$ flessi ascendenti

$\left(2\pi - \arccos\frac{1}{4} + 2k\pi\,;\,-\frac{3\sqrt{15}}{8}\right)$ flessi discendenti


f(x) = 2 sin(x) - sin(2 x)
O
A
B
M
N
F2
F4
-8
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
5
4
3
2
1
0
-1
-2
-3
-4
-5

$$y = \frac{\cos x}{1 - \cos x}$$

dominio:

$1 - \cos x \neq 0 \rightarrow \cos x \neq 1$

$x \neq 2k\pi$

$D = \{ x \in \mathbb{R} \mid x \neq 2k\pi, k \in \mathbb{Z} \}$

simmetrie:

$f(-x) = \frac{\cos x}{1 - \cos x} = f(x)$

$f$ è pari

periodicità:

$T = 2\pi$

intersezioni con gli assi:

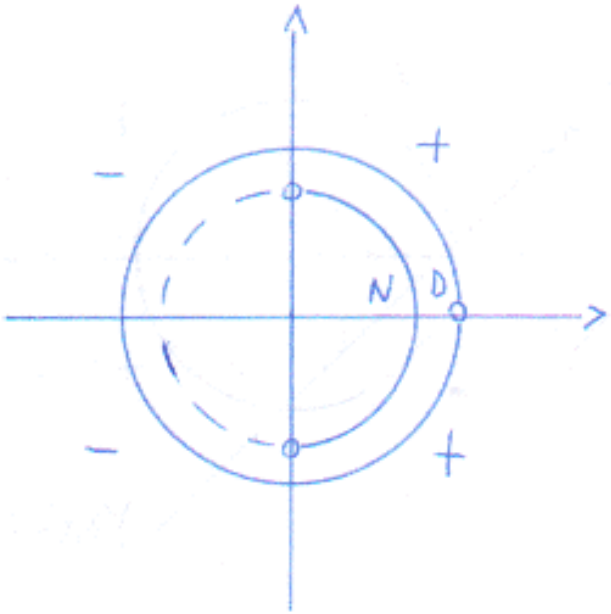
$0 \notin D$

$\begin{cases} y = 0 \\ \frac{\cos x}{1 - \cos x} = 0 \end{cases}$ $\quad x = \frac{\pi}{2} + k\pi$

$A\left(\frac{\pi}{2}; 0\right) \quad B\left(\frac{3}{2}\pi; 0\right)$ etc.

segno:

$f(x) > 0 \qquad \dfrac{\cos x}{1-\cos x} > 0$




$$I.P. = \left\{ x \in \mathbb{R} \;\middle|\; 2K\pi < x < \frac{\pi}{2} + 2K\pi \;\vee\; \frac{3}{2}\pi + 2K\pi < x < 2\pi + 2K\pi \right\}$$

$$I.N. = \left\{ x \in \mathbb{R} \;\middle|\; \frac{\pi}{2} + 2K\pi < x < \frac{3}{2}\pi + 2K\pi \right\}$$

asintoti:

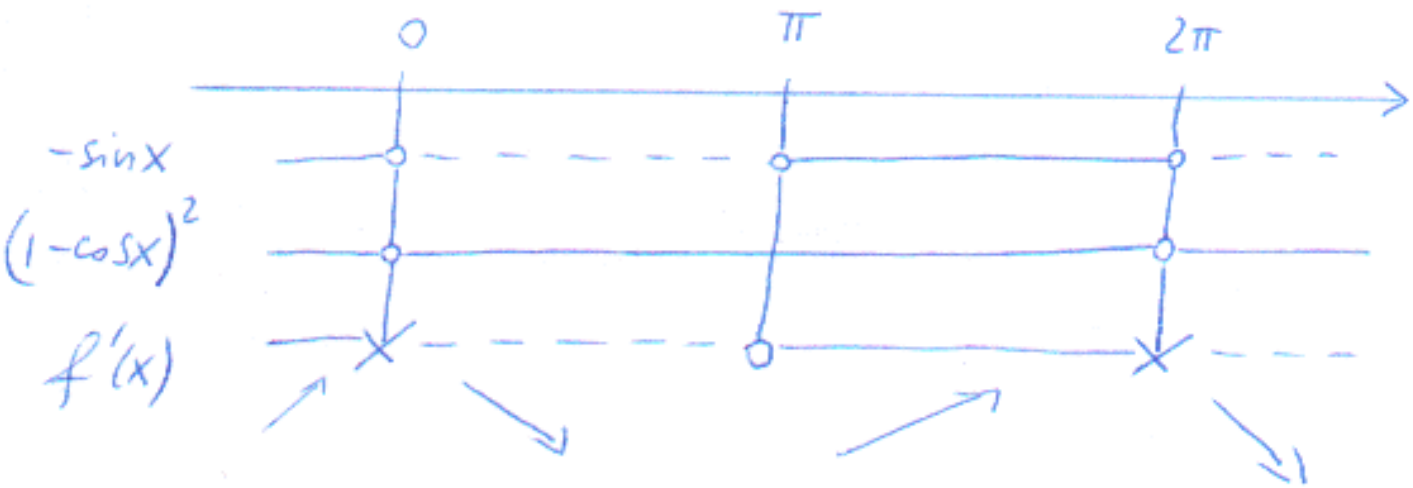
$$\lim_{x \to 2K\pi} \frac{\cos x}{1-\cos x} = \frac{1}{0^+} = +\infty$$

$x = 2K\pi$ asintoti verticali

massimi e minimi:

$$f'(x) = \frac{-\sin x + \cancel{\sin x \cos x} - \cancel{\sin x \cos x}}{(1-\cos x)^2}$$

$$f'(x) = \frac{-\sin x}{(1-\cos x)^2}$$




$$\left(\pi + 2k\pi \,;\, -\frac{1}{2}\right) \quad \text{minimi}$$

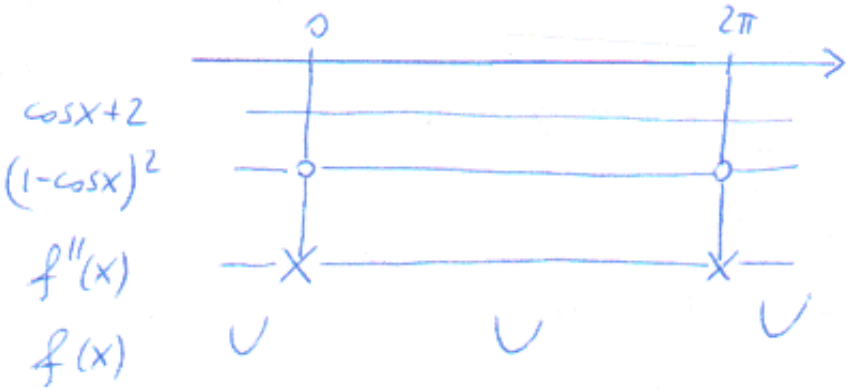
flessi :

$$f''(x) = \frac{-\cos x\,(1-\cos x)^2 + 2\,(1-\cos x)\sin^2 x}{(1-\cos x)^4}$$

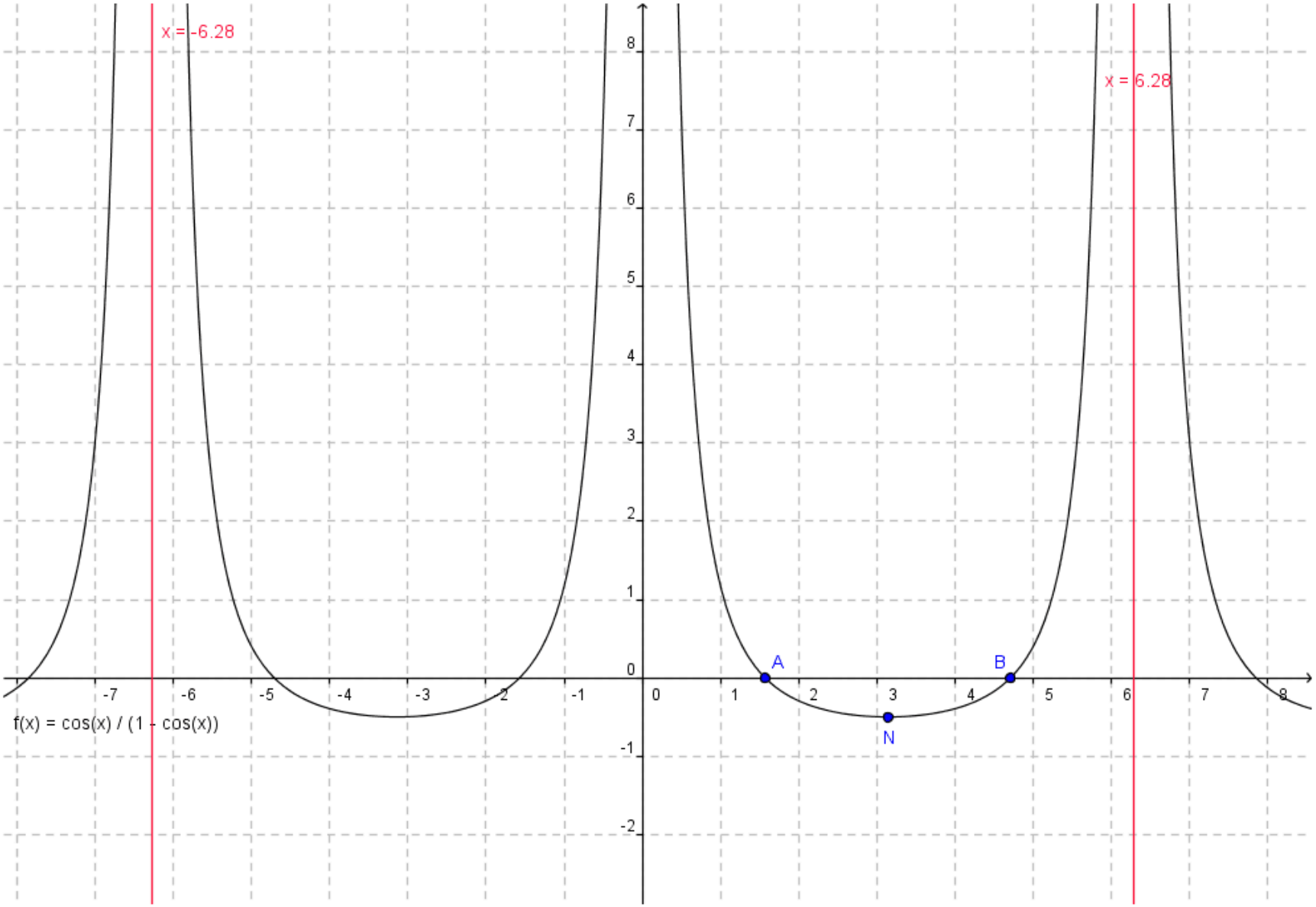
$$f''(x) = \frac{\cancel{(1-\cos x)}\,(-\cos x + \cos^2 x + 2 - 2\cos^2 x)}{(1-\cos x)^{\cancel{4}\,3}}$$

$$f''(x) = \frac{-\cos^2 x - \cos x + 2}{(1-\cos x)^3} = \frac{\cancel{(1-\cos x)}\,(\cos x + 2)}{(1-\cos x)^{\cancel{3}\,2}}$$

$$f''(x) = \frac{\cos x + 2}{(1-\cos x)^2}$$




Non ci sono flessi


x = -6.28
x = 6.28
A
B
N
f(x) = cos(x) / (1 - cos(x))

$$\boxed{y = \arcsin\sqrt{2x - x^2}}$$

dominio:

$$\begin{cases} 2x - x^2 \geq 0 \\ -1 \leq \sqrt{2x - x^2} \leq 1 \end{cases}$$

$$\begin{cases} x(2-x) \geq 0 \\ x(2-x) \leq 1 \end{cases}$$

$$\begin{cases} 0 \leq x \leq 2 \\ x^2 - 2x + 1 \geq 0 \end{cases}$$

$$\begin{cases} 0 \leq x \leq 2 \\ \forall x \in \mathbb{R} \end{cases}$$

$$D = [0; 2]$$

intersezioni con gli assi:

$$\begin{cases} x = 0 \\ y = 0 \end{cases} \qquad O(0;0)$$

$$\begin{cases} y = 0 \\ \arcsin\sqrt{2x - x^2} = 0 \end{cases}$$

$$\sqrt{2x - x^2} = 0$$

$$x(2-x) = 0 \qquad x_1 = 0 \quad x_2 = 2$$

$$A(2;0) \qquad O(0;0)$$

segno:

$f(x) > 0 \rightarrow \arcsin\sqrt{2x-x^2} > 0$

$0 < \sqrt{2x-x^2} \leq 1$

$0 < 2x-x^2 \leq 1$

$\begin{cases} 2x-x^2 > 0 \\ x^2-2x+1 \geq 0 \end{cases} \rightarrow \begin{cases} 0 < x < 2 \\ \forall x \in \mathbb{R} \end{cases}$

I.P. $= (0;2)$

I.N. $= \emptyset$

asintoti:

$\lim_{x \to 0^+} f(x) = 0$

$\lim_{x \to 2^-} f(x) = 0$

Non ci sono asintoti

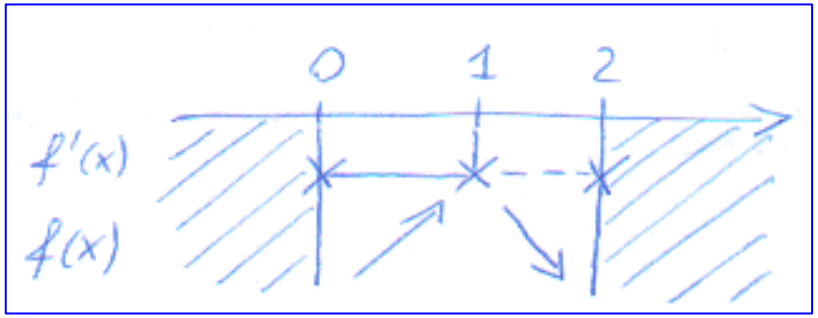
massimi e minimi:

$$f'(x) = \frac{2-2x}{2\sqrt{2x-x^2}} \cdot \frac{1}{\sqrt{1-2x+x^2}}$$

$$f'(x) = \frac{\cancel{2}(1-x)}{\cancel{2}\sqrt{2x-x^2}\,\sqrt{(1-x)^2}}$$

$$f'(x) = \frac{1-x}{|1-x|\,\sqrt{2x-x^2}}$$

$$f'(x) = \begin{cases} \dfrac{1}{\sqrt{2x-x^2}} & \text{per } 0<x<1 \\ -\dfrac{1}{\sqrt{2x-x^2}} & \text{per } 1<x<2 \end{cases}$$




Non ci sono punti stazionari

$\left(1;\ \frac{\pi}{2}\right)$ massimo (punto angoloso)

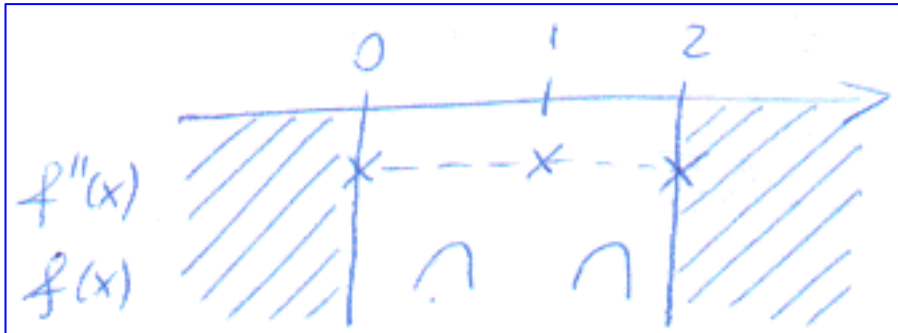
$(0;0)$ minimo

$(2;0)$ minimo

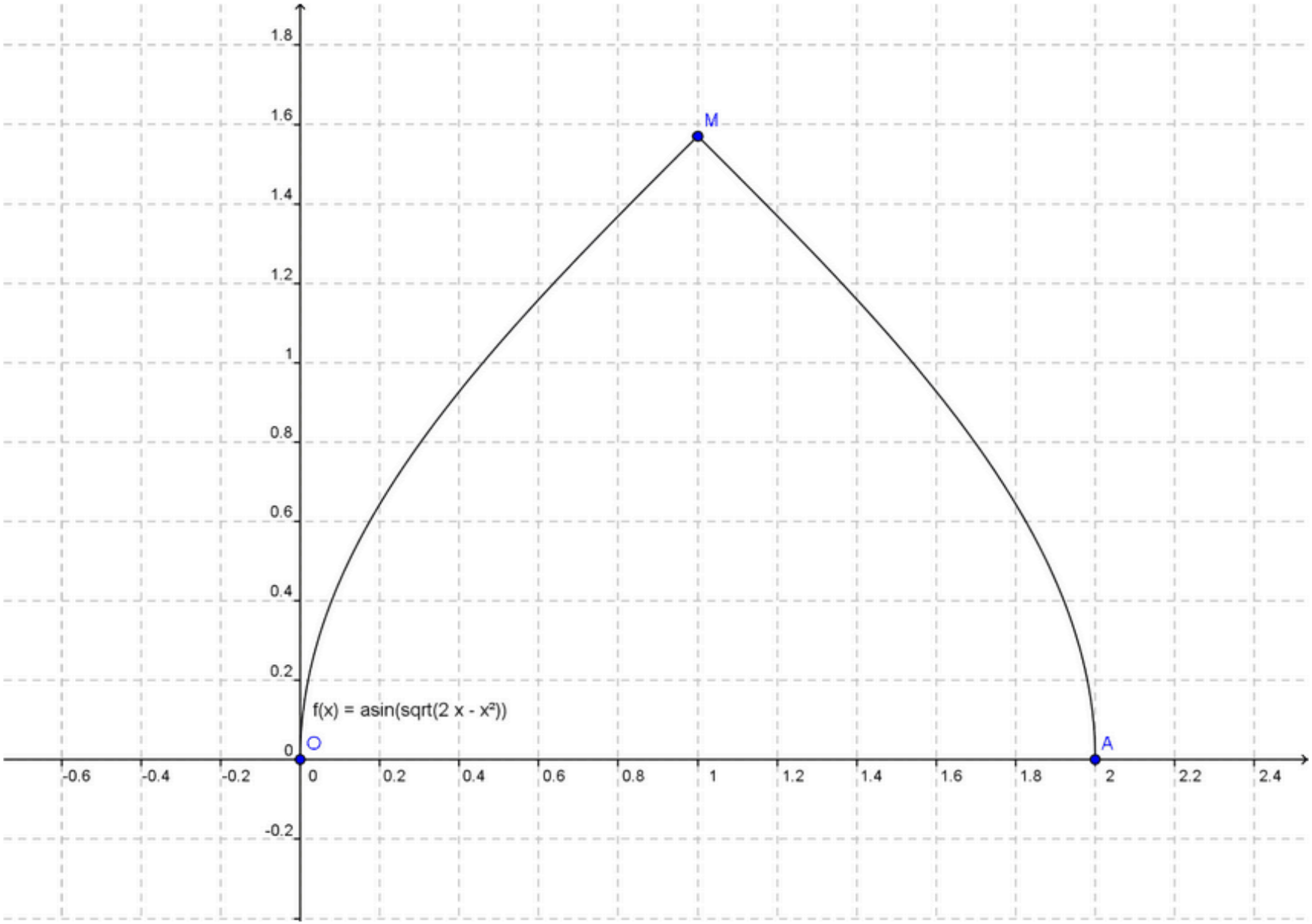
flessi:

$$f''(x) = \begin{cases} -\frac{1}{2}\,\dfrac{2-2x}{\sqrt{(2x-x^2)^3}} & 0<x<1 \\ +\frac{1}{2}\,\dfrac{2-2x}{\sqrt{(2x-x^2)^3}} & 1<x<2 \end{cases}$$

$$f''(x) = \begin{cases} \dfrac{x-1}{\sqrt{(2x-x^2)^3}} & 0<x<1 \\ \dfrac{1-x}{\sqrt{(2x-x^2)^3}} & 1<x<2 \end{cases}$$




Non ci sono flessi


M
O
A
f(x) = asin(sqrt(2 x - x²))
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
2.2
2.4

$$y = \operatorname{arc}tg \frac{1}{1-x}$$

dominio :

$1-x \neq 0 \rightarrow x \neq 1$

$D = \mathbb{R} - \{1\}$

intersezioni con gli assi :

$$\begin{cases} x=0 \\ y=\frac{\pi}{4} \end{cases} \qquad A\left(0; \frac{\pi}{4}\right)$$

$$\begin{cases} y=0 \\ \operatorname{arctg} \frac{1}{1-x} = 0 \end{cases}$$

$\frac{1}{1-x} = 0$ impossi.

Non ci sono intersezioni con l'asse x

segno:

$f(x) > 0$

$\operatorname{arctg} \frac{1}{1-x} > 0$

$\frac{1}{1-x} > 0 \rightarrow x < 1$

$I.P. = (-\infty; 1)$

$I.N. = (1; +\infty)$

asintoti:

$$\lim_{x \to 1^-} \operatorname{arctg} \frac{1}{1-x} = \frac{\pi}{2}$$

$$\lim_{x \to 1^+} \operatorname{arctg} \frac{1}{1-x} = -\frac{\pi}{2}$$

Non ci sono asintoti verticali

$$\lim_{x \to \infty} \operatorname{arctg} \frac{1}{1-x} = 0$$

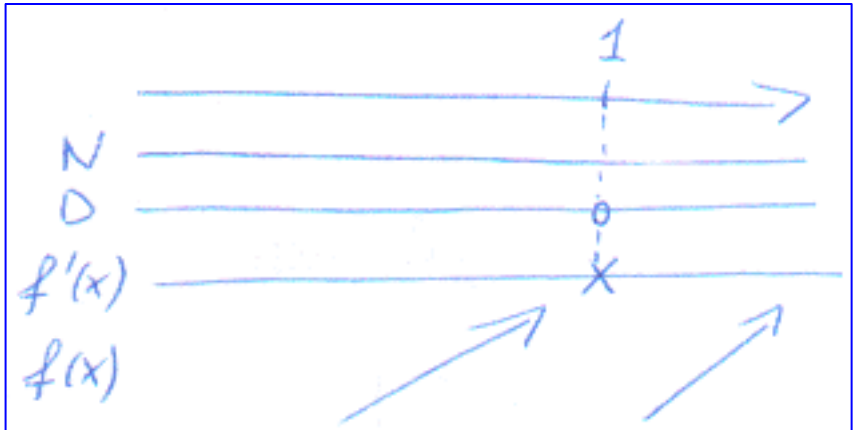
$y=0$ asintoto orizzontale

massimi e minimi:

$$f'(x) = \frac{1}{(1-x)^2} \cdot \frac{1}{1+\left(\frac{1}{1-x}\right)^2}$$

$$f'(x) = \frac{1}{1-2x+x^2+1} = \frac{1}{x^2-2x+2}$$

$f'(x) > 0$

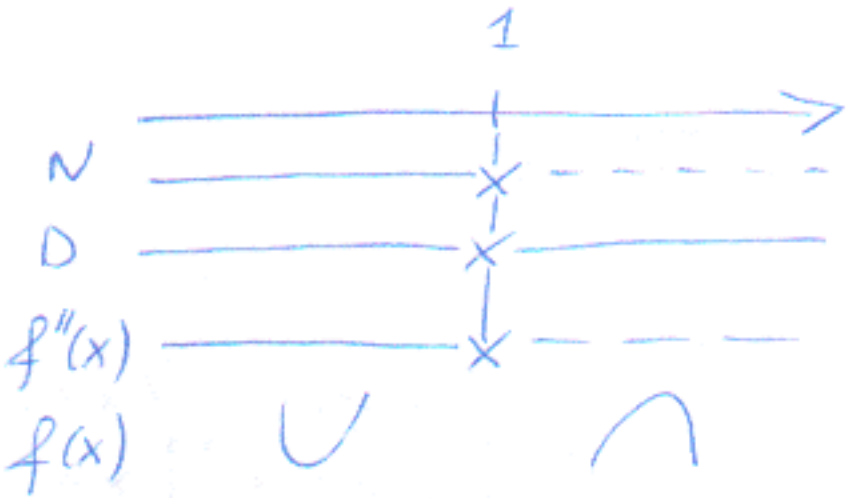



Non ci sono massimi e minimi

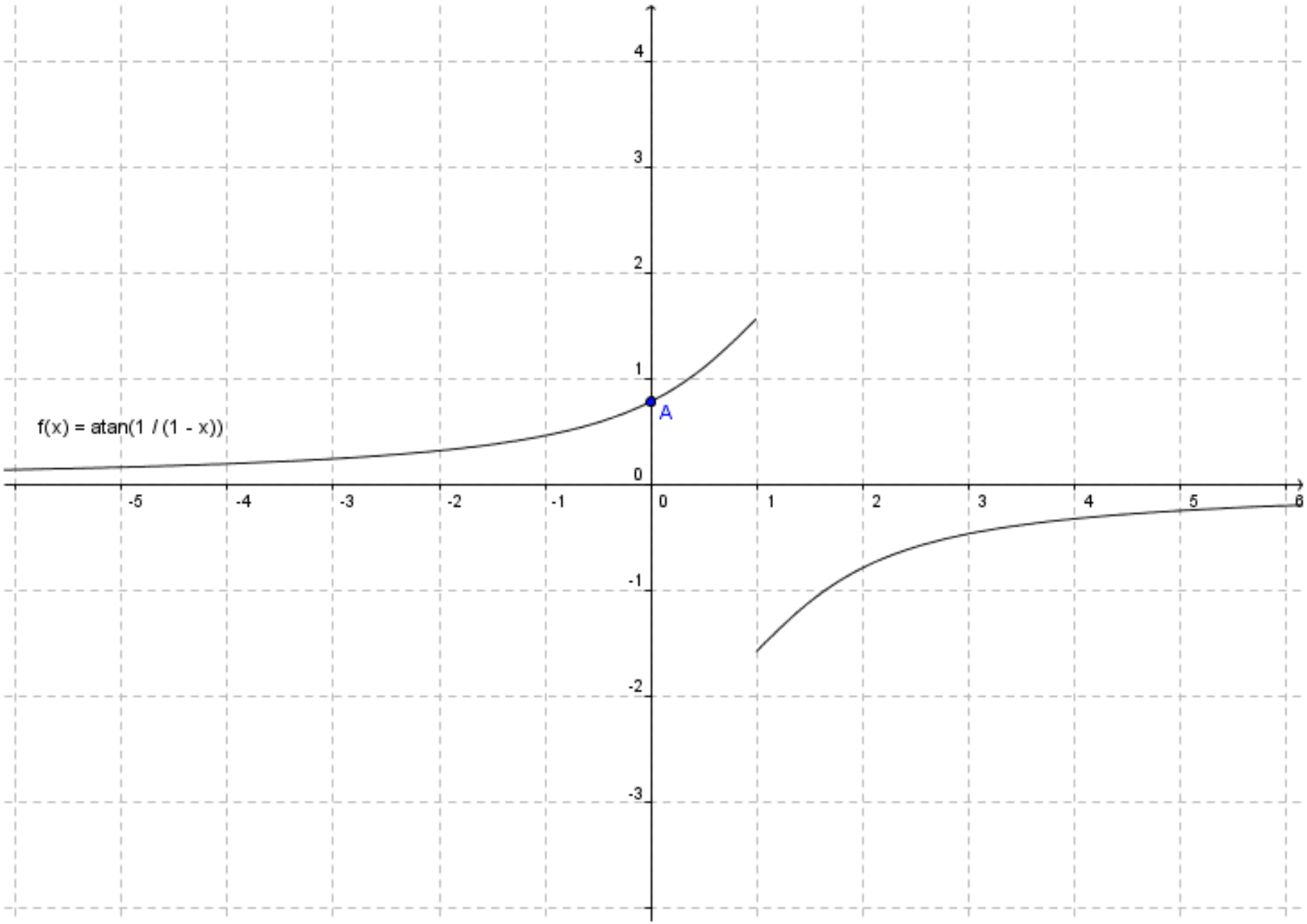
flessi:

$$f''(x) = -\frac{(2x-2)}{(x^2-2x+2)^2} = \frac{2(1-x)}{(x^2-2x+2)^2} \qquad x \neq 1$$

$f''(x) > 0$




Non ci sono flessi


f(x) = atan(1 / (1 - x))
A
4
3
2
1
0
-1
-2
-3
-5
-4
-3
-2
-1
0
1
2
3
4
5
6

# Integrali indefiniti

# Integrali immediati

$$\int [f(x)]^{\alpha} f'(x)\,dx = \frac{[f(x)]^{\alpha+1}}{\alpha+1} + c, \quad \text{con } \alpha \neq -1$$

$$\int \frac{x}{\sqrt{1+x^2}}\,dx = \int (1+x^2)^{-\frac{1}{2}} x\,dx = \frac{1}{2}\int (1+x^2)^{-\frac{1}{2}} \cdot 2x\,dx = \frac{1}{2}\,\frac{(1+x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c = \sqrt{1+x^2} + c$$

$$\int \frac{dx}{x \log^2 x} = \int [\log x]^{-2} \frac{1}{x}\,dx = -[\log x]^{-1} + c = -\frac{1}{\log x} + c$$

$$\int f'(x) \cos f(x)\,dx = \operatorname{sen} f(x) + c$$

$$\int \frac{\cos \log x}{x}\,dx = \int \cos(\log x) \cdot \frac{1}{x}\,dx = \operatorname{sen} \log x + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right| + c$$

$$\int \frac{e^x}{1-2e^x}\,dx = -\frac{1}{2}\int \frac{-2e^x}{1-2e^x}\,dx = -\frac{1}{2}\log\left|1-2e^x\right| + c$$

$$\int \frac{dx}{1+e^x} = \int \frac{1+e^x-e^x}{1+e^x}\,dx = \int \frac{\cancel{1+e^x}^{\,1}}{\cancel{1+e^x}_{\,1}}\,dx - \int \frac{e^x}{1+e^x}\,dx = x - \log\left(1+e^x\right) + c$$

$$\int \frac{2-\sin x}{2x+\cos x}\,dx = \log\left|2x+\cos x\right| + c$$

$$\int \frac{2x+1}{x+2}\,dx = \int \frac{2x+4-3}{x+2}\,dx = \int \frac{2\cancel{(x+2)}^{\,1}}{\cancel{x+2}_{\,1}}\,dx - \int \frac{3}{x+2}\,dx =$$

$$= 2\int dx - 3\int \frac{1}{x+2}\,dx = 2x - 3\log\left|x+2\right| + c$$

$$\int f'(x)\, e^{f(x)}\, dx = e^{f(x)} + c$$

$$\int e^{x+\sin x}(1+\cos x)\,dx = e^{x+\sin x} + C$$

$$\int \frac{f'(x)}{\sqrt{1-[f(x)]^2}}\, dx = \operatorname{arcsen} f(x) + c$$

$$\int \frac{dx}{\sqrt{1-8x^2}} = \frac{1}{\sqrt{8}} \int \frac{\sqrt{8}\,dx}{\sqrt{1-8x^2}} = \frac{1}{2\sqrt{2}} \arcsin(\sqrt{8}x) + C = \frac{\sqrt{2}}{4} \arcsin(\sqrt{8}x) + C$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}\, dx = \frac{1}{a} \operatorname{arctg} \frac{f(x)}{a} + c.$$

$$\int \frac{2}{3+x^2}\,dx = 2\int \frac{1}{(\sqrt{3})^2+x^2}\,dx = \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{x}{\sqrt{3}} + C = \frac{2}{3}\sqrt{3} \operatorname{arctg} \frac{x\sqrt{3}}{3} + C$$

$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}}\, dx = \text{arcsen}\, \frac{f(x)}{|a|} + c$$

$$\int \frac{dx}{\sqrt{4-25x^2}} = \frac{1}{5} \int \frac{5}{\sqrt{4-25x^2}}\, dx = \frac{1}{5} \arcsin \frac{5x}{2} + C$$

# Integrali funzioni razionali fratte

$$\int \frac{dx}{x^2-3x+2}$$

$$f(x) = \frac{1}{x^2-3x+2} = \frac{1}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1} = \frac{Ax-A+Bx-2B}{(x-2)(x-1)} = \frac{(A+B)x-A-2B}{(x-2)(x-1)}$$

$$\begin{cases} A+B=0 \\ -A-2B=1 \end{cases} \qquad \begin{cases} B=-A \\ -A+2A=1 \end{cases} \qquad \begin{cases} B=-1 \\ A=1 \end{cases}$$

$$f(x) = \frac{1}{x-2} - \frac{1}{x-1}$$

$$\int \frac{dx}{x^2-3x+2} = \int \frac{dx}{x-2} - \int \frac{dx}{x-1} = \log|x-2| - \log|x-1| + c = \log\left|\frac{x-2}{x-1}\right| + c$$

$$\int \frac{dx}{x^2-x-6}$$

$$f(x) = \frac{1}{x^2-x-6} = \frac{1}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2} = \frac{Ax+2A+Bx-3B}{(x-3)(x+2)} = \frac{(A+B)x+2A-3B}{(x-3)(x+2)}$$

$$\begin{cases} A+B=0 \\ 2A-3B=1 \end{cases} \qquad \begin{cases} B=-A \\ 5A=1 \end{cases} \qquad \begin{cases} B=-\frac{1}{5} \\ A=\frac{1}{5} \end{cases} \qquad f(x) = \frac{1}{5}\,\frac{1}{x-3} - \frac{1}{5}\,\frac{1}{x+2}$$

$$\int \frac{dx}{x^2-x-6} = \frac{1}{5}\int \frac{dx}{x-3} - \frac{1}{5}\int \frac{dx}{x+2} = \frac{1}{5}\log|x-3| - \frac{1}{5}\log|x+2| + C = \frac{1}{5}\log\left|\frac{x-3}{x+2}\right| + C$$

$$\int \frac{2}{x^2-3}\,dx$$

$$f(x) = \frac{2}{x^2-3} = \frac{2}{(x+\sqrt{3})(x-\sqrt{3})} = \frac{A}{x+\sqrt{3}} + \frac{B}{x-\sqrt{3}} = \frac{Ax-A\sqrt{3}+Bx+B\sqrt{3}}{(x+\sqrt{3})(x-\sqrt{3})} = \frac{(A+B)x-A\sqrt{3}+B\sqrt{3}}{(x+\sqrt{3})(x-\sqrt{3})}$$

$$\begin{cases} A+B=0 \\ -A\sqrt{3}+B\sqrt{3}=2 \end{cases} \qquad \begin{cases} A=-B \\ 2B\sqrt{3}=2 \end{cases} \qquad \begin{cases} A=-\frac{1}{\sqrt{3}} \\ B=\frac{1}{\sqrt{3}} \end{cases} \qquad f(x) = -\frac{\sqrt{3}}{3}\,\frac{1}{x+\sqrt{3}} + \frac{\sqrt{3}}{3}\,\frac{1}{x-\sqrt{3}}$$

$$\int \frac{2}{x^2-3}\,dx = -\frac{\sqrt{3}}{3}\int \frac{dx}{x+\sqrt{3}} + \frac{\sqrt{3}}{3}\int \frac{dx}{x-\sqrt{3}} = -\frac{\sqrt{3}}{3}\log|x+\sqrt{3}| + \frac{\sqrt{3}}{3}\log|x-\sqrt{3}| + C = \frac{\sqrt{3}}{3}\log\left|\frac{x-\sqrt{3}}{x+\sqrt{3}}\right| + C$$

$$\int \frac{dx}{x^2+4x+5} = \int \frac{dx}{(x+2)^2+1} = \operatorname{arctg}(x+2) + C$$

$$\int \frac{dx}{x^2+8x+16} = \int \frac{dx}{(x+4)^2} = -\frac{1}{x+4} + C$$

$$\int \frac{x+1}{x^2-2x+1}\,dx = \int \frac{x+1}{(x-1)^2}\,dx = \int \frac{x-1+2}{(x-1)^2}\,dx = \int \frac{dx}{x-1} + \int \frac{2}{(x-1)^2}\,dx = \log|x-1| - \frac{2}{x-1} + C$$

$$\int \frac{2x+1}{x^2+25}\,dx = \int \frac{2x}{x^2+25}\,dx + \int \frac{1}{x^2+25}\,dx = \log(x^2+25) + \frac{1}{5}\operatorname{arctg}\frac{x}{5} + C$$

$$\int \frac{2x+5}{x^2+3x+5}\,dx = \int \frac{2x+3+2}{x^2+3x+5}\,dx = \int \frac{2x+3}{x^2+3x+5}\,dx + 2\int \frac{1}{\left(x+\frac{3}{2}\right)^2+\frac{11}{4}}\,dx =$$

$$= \log\left(x^2+3x+5\right) + \frac{4}{\sqrt{11}}\operatorname{arctg}\frac{2x+3}{\sqrt{11}} + C$$

# Metodo integrazione per sostituzione

$$\int f(x) = \int f\left[g(t)\right]g'(t)dt$$

$$\int e^{3x-1}dx$$

Ponendo

$$t = 3x - 1$$

ricaviamo x:

$$x = \frac{1+t}{3}$$

e quindi

$$\frac{dx}{dt} = \frac{1}{3} \rightarrow dx = \frac{1}{3}dt$$

Ora, ritornando all'integrale iniziale:

$$\int e^{3x-1}dx = \int e^{t} \cdot \frac{1}{3}dt$$

$$\int e^{3x-1}dx = \frac{1}{3}\int e^{t}dt$$

$$\int e^{3x-1}dx = \frac{1}{3}e^{t} + C$$

Risulta quindi:

$$\int e^{3x-1}dx = \frac{1}{3}e^{3x-1} + C$$

$$\int \frac{e^x}{e^x+2}dx$$

Ponendo

$$t = e^x$$

ricaviamo x:

$$x = \ln t$$

e quindi

$$\frac{dx}{dt} = \frac{1}{t} \rightarrow dx = \frac{1}{t}dt$$

Ora, ritornando all'integrale iniziale:

$$\int \frac{e^x}{e^x+2}dx = \int \frac{t \cdot \frac{1}{t}}{t+2}dt$$

$$\int \frac{e^x}{e^x+2}dx = \int \frac{1}{t+2}dt$$

$$\int \frac{e^x}{e^x+2}dx = \ln|t+2| + C$$

Risulta quindi:

$$\int \frac{e^x}{e^x+2}dx = \ln\left(e^x+2\right) + C$$

$$\int \frac{\ln x}{x} dx$$

Ponendo

$$t = \ln x$$

ricaviamo x:

$$x = e^t$$

e quindi

$$\frac{dx}{dt} = e^t \rightarrow dx = e^t dt$$

Ora, ritornando all'integrale iniziale:

$$\int \frac{\ln x}{x} dx = \int \frac{t}{e^t} e^t dt$$

$$\int \frac{\ln x}{x} dx = \int t dt$$

$$\int \frac{\ln x}{x} dx = \frac{t^2}{2} + C$$

Risulta quindi:

$$\int \frac{\ln x}{x} dx = \frac{\ln^2 x}{2} + C$$

$$\boxed{\int \frac{1}{1+e^x}dx}$$

Ponendo

$$t = e^x$$

ricaviamo x:

$$x = \ln t$$

e quindi

$$\frac{dx}{dt} = \frac{1}{t} \rightarrow dx = \frac{1}{t}dt$$

Ora, ritornando all'integrale iniziale:

$$\int \frac{1}{1+e^x}dx = \int \frac{1}{1+t} \cdot \frac{1}{t}dt$$

$$\int \frac{1}{1+e^x}dx = \int \frac{1}{t(t+1)}dt$$

L'integrale diventa:

$$\int \frac{1}{1+e^x}dx = \int \frac{1}{t}dt - \int \frac{1}{t+1}dt$$

$$\int \frac{1}{1+e^x}dx = \ln|t| - \ln|t+1| + C$$

Risulta quindi:

$$\int \frac{1}{1+e^x}dx = \ln e^x - \ln\left(e^x+1\right) + C$$

$$\int \frac{1}{1+e^x}dx = x - \ln\left(e^x+1\right) + C$$

$$\boxed{\int \sqrt{1-x^2}dx}$$

Ponendo

$$x = \sin t$$

ricaviamo t:

$$t = \arcsin x$$

e quindi

$$\frac{dx}{dt} = \cos t \rightarrow dx = \cos t dt$$

Ora, ritornando all'integrale iniziale:

$$\int \sqrt{1-x^2}dx = \int \sqrt{1-\sin^2 t}\cos t dt$$

$$\int \sqrt{1-x^2}dx = \int \cos^2 t dt$$

Ricordando la formula di riduzione del coseno:

$$\cos^2 t = \frac{1 + \cos 2t}{2}$$

abbiamo che

$$\int \sqrt{1 - x^2} dx = \frac{1}{2}\left(\int dt + \int \cos 2t dt\right)$$

$$\int \sqrt{1 - x^2} dx = \frac{1}{2}\left(t + \frac{1}{2}\sin 2t\right) + C$$

$$\int \sqrt{1 - x^2} dx = \frac{1}{2}(t + \sin t \cos t) + C$$

$$\int \sqrt{1 - x^2} dx = \frac{1}{2}\left(t + \sin t\sqrt{1 - \sin^2 t}\right) + C$$

Risulta quindi:

$$\int \sqrt{1 - x^2} dx = \frac{1}{2}\left(\arcsin x + x\sqrt{1 - x^2}\right) + C$$

$$\boxed{\int \frac{dx}{x-\sqrt{x+2}}}$$

Ponendo

$$t = \sqrt{x+2}$$

ricaviamo x:

$$x = t^2 - 2$$

e quindi

$$\frac{dx}{dt} = 2t \rightarrow dx = 2tdt$$

Ora, ritornando all'integrale iniziale:

$$\int \frac{dx}{x-\sqrt{x+2}} = \int \frac{2tdt}{t^2-2-t}$$

$$\int \frac{dx}{x-\sqrt{x+2}} = \int \frac{2t}{(t+1)(t-2)}dt$$

Ricaviamo A e B tali che

$$\frac{2t}{(t+1)(t-2)} = \frac{A}{t+1} + \frac{B}{t-2}$$

$$\frac{2t}{(t+1)(t-2)} = \frac{(A+B)t - 2A + B}{(t+1)(t-2)}$$

$$A = \frac{2}{3}\ ;\ B = \frac{4}{3}$$

L'integrale diventa:

$$\int \frac{dx}{x - \sqrt{x+2}} = \frac{2}{3}\int \frac{1}{t+1}dt + \frac{4}{3}\int \frac{1}{t-2}dt$$

$$\int \frac{dx}{x - \sqrt{x+2}} = \frac{2}{3}\ln|t+1| + \frac{4}{3}\ln|t-2| + C$$

Risulta quindi:

$$\int \frac{dx}{x - \sqrt{x+2}} = \frac{2}{3}\ln\left(\sqrt{x+2}+1\right) + \frac{4}{3}\ln\left|\sqrt{x+2}-2\right| + C$$

$$\int \frac{1}{\sqrt{1+2x^2}}\,dx$$

$$\sqrt{1+2x^2} = t - \sqrt{2}\,x \qquad t = \sqrt{2}\,x + \sqrt{1+2x^2}$$

$$1+2x^2 = t^2 - 2\sqrt{2}\,t\,x + 2x^2$$

$$2\sqrt{2}\,t\,x = t^2 - 1$$

$$x = \frac{t^2-1}{2\sqrt{2}\,t} \qquad dx = \frac{4\sqrt{2}\,t^2 - 2\sqrt{2}\,t^2 + 2\sqrt{2}}{(2\sqrt{2}\,t)^2}\,dt = \frac{t^2+1}{2\sqrt{2}\,t^2}\,dt$$

$$\sqrt{1+2x^2} = t - \frac{t^2-1}{2t} = \frac{t^2+1}{2t}$$

$$\int \frac{1}{\sqrt{1+2x^2}}\,dx = \int \frac{2t}{\cancel{t^2+1}}\,\frac{\cancel{t^2+1}}{2\sqrt{2}\,t^2}\,dt = \int \frac{dt}{\sqrt{2}\,t} = \frac{1}{\sqrt{2}}\log|t| + c = \frac{1}{\sqrt{2}}\log\left(\sqrt{2}\,x + \sqrt{1+2x^2}\right) + c$$

$$\int \frac{dx}{\operatorname{cotg}^3 x\,\sin^2 x} = -\int \frac{dt}{t^3} = \frac{1}{2t^2} + c = \frac{1}{2}\operatorname{tg}^2 x + c$$

$$t = \operatorname{cotg} x$$

$$dt = -\frac{1}{\sin^2 x}\,dx$$

$$\int tg^3 x\,dx =$$

$$t = tg\,x$$

$$x = arctg\,t$$

$$dx = \frac{1}{1+t^2}\,dt$$

$$= \int \frac{t^3}{1+t^2}\,dt =$$

$$= \int \frac{t^3+t-t}{1+t^2}\,dt =$$

$$= \int \frac{t(t^2+1)}{1+t^2}\,dt - \int \frac{t}{1+t^2}\,dt = \int t\,dt - \frac{1}{2}\int \frac{2t}{1+t^2}\,dt = \frac{t^2}{2} - \frac{1}{2}\log(1+t^2) + C =$$

$$= \frac{1}{2}\,tg^2 x - \log\sqrt{1+tg^2 x} + C$$

$$\int \frac{\cos x}{1-\cos x}\,dx =$$

$$t = tg\,\frac{x}{2}$$

$$x = 2\,arctg\,t$$

$$dx = \frac{2\,dt}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$= \int \frac{1-t^2}{1+t^2}\,\frac{1}{1-\frac{1-t^2}{1+t^2}}\,\frac{2\,dt}{1+t^2} =$$

$$= \int \frac{1-t^2}{1+t^2}\,\frac{2\,dt}{2t^2} = \int \frac{1+t^2-2t^2}{t^2(1+t^2)}\,dt = \int \frac{\cancel{1+t^2}}{t^2\cancel{(1+t^2)}}\,dt - \int \frac{2\cancel{t^2}}{\cancel{t^2}(1+t^2)}\,dt =$$

$$= -\frac{1}{t} - 2\,arctg\,t + C = -cotg\,\frac{x}{2} - x + C$$

$$\int \sqrt{9-x^2}\,dx \qquad x = 3\sin t \qquad dx = 3\cos t\,dt \qquad t = \arcsin\frac{x}{3}$$

$$\sqrt{9-x^2} = \sqrt{9-9\sin^2 t} = 3\sqrt{\cos^2 t} = 3\cos t$$

$$\int \sqrt{9-x^2}\,dx = 9\int \cos^2 t\,dt = 9\int \frac{1+\cos 2t}{2}\,dt = 9\left(\frac{1}{2}t + \frac{1}{4}\sin 2t\right) + C =$$

$$= \frac{9}{2}\arcsin\frac{x}{3} + \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C = \frac{9}{2}\arcsin\frac{x}{3} + \frac{x}{2}\sqrt{9-x^2} + C$$

$$\int \sqrt{e^x-1}\,dx \qquad t = \sqrt{e^x-1} \qquad t^2 = e^x - 1 \qquad e^x = t^2+1$$

$$x = \log(t^2+1)$$

$$dx = \frac{2t}{t^2+1}\,dt$$

$$\int \sqrt{e^x-1}\,dx = \int \frac{2t^2}{t^2+1}\,dt = \int \frac{2t^2+2-2}{t^2+1}\,dt = 2\int dt - 2\int \frac{dt}{t^2+1} =$$

$$= 2t - 2\,\mathrm{arctg}\,t + C = 2\sqrt{e^x-1} - 2\,\mathrm{arctg}\sqrt{e^x-1} + C$$

$$\int \frac{x}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\int \frac{dt}{\sqrt{1-t^2}} = \frac{1}{2}\arcsin t + C = \frac{1}{2}\arcsin x^2 + C$$

$t = x^2$

$dt = 2x\,dx$

$$\int \frac{dx}{e^x + e^{-x}} = \int \frac{1}{t+\frac{1}{t}}\cdot\frac{dt}{t} = \int \frac{dt}{t^2+1} = \operatorname{arctg} t + C = \operatorname{arctg} e^x + C$$

$t = e^x$

$dt = e^x\,dx$

$dx = \frac{dt}{t}$

$$\int \frac{dx}{x\log^4 x} = \int \frac{dt}{t^4} = -\frac{1}{3t^3} + C = -\frac{1}{3\log^3 x} + C$$

$t = \log x$

$dt = \frac{1}{x}\,dx$

# Metodo integrazione per parti

$$\int f(x)\cdot g'(x)\,dx = f(x)\cdot g(x) - \int f'(x)\cdot g(x)\,dx$$

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x + C = e^x(x-1) + C$$

$f'(x) = e^x \qquad f(x) = e^x$

$g(x) = x \qquad g'(x) = 1$

$$\int x^2 \log x\, dx = \frac{x^3}{3}\log x - \frac{1}{3}\int x^3 \frac{1}{x}\, dx = \frac{x^3}{3}\log x - \frac{1}{3}\int x^2 dx = \frac{x^3}{3}\log x - \frac{1}{9}x^3 + C =$$

$$= \frac{x^3}{3}\left(\log x - \frac{1}{3}\right) + C$$

$f'(x) = x^2 \qquad f(x) = \frac{x^3}{3}$

$g(x) = \log x \qquad g'(x) = \frac{1}{x}$

$$\int x \sin x \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + C$$

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C$$

$$\int x^2 \cos x \, dx = x^2 \sin x - 2 \int x \sin x \, dx = x^2 \sin x + 2x \cos x - 2 \int \cos x \, dx =$$

$$= x^2 \sin x + 2x \cos x - 2 \sin x + C$$

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

$$2 \int e^x \sin x \, dx = e^x (\sin x - \cos x) \quad \rightarrow \quad \int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C$$

$$\int \arcsin x \, dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \, dx = x \arcsin x + \sqrt{1-x^2} + C$$

$$\int \cos^3 x\,dx = \int \cos x \cdot \cos^2 x\,dx = \sin x \cos^2 x - \int \sin x \cdot (-2\cos x \sin x)\,dx =$$

$$= \sin x\left(1-\sin^2 x\right) + 2\int \sin^2 x \cos x\,dx = \sin x - \sin^3 x + 2\int \left(1-\cos^2 x\right)\cos x\,dx =$$

$$= \sin x - \sin^3 x + 2\sin x - 2\int \cos^3 x\,dx$$

$$3\int \cos^3 x\,dx = 3\sin x - \sin^3 x \longrightarrow \int \cos^3 x\,dx = \sin x - \frac{1}{3}\sin^3 x + C$$

$$\int \log(1+x)\,dx = x\log(1+x) - \int \frac{x}{1+x}\,dx = x\log(1+x) - \int \frac{1+x-1}{1+x}\,dx = x\log(1+x) - x + \log(1+x) + C =$$

$$= (1+x)\log(1+x) - x + C$$

$$\int \frac{\log x}{x^3}\,dx = -\frac{1}{2x^2}\log x + \int \frac{1}{2x^2}\cdot\frac{1}{x}\,dx = -\frac{1}{2x^2}\log x - \frac{1}{4x^2} + C = -\frac{2\log x + 1}{4x^2} + C$$

$$\int x \log(1+4x^2)\,dx = \frac{x^2}{2}\log(1+4x^2) - \int \frac{x^2}{2}\,\frac{8x}{1+4x^2}\,dx =$$

$$= \frac{x^2}{2}\log(1+4x^2) - \int \frac{4x^3}{1+4x^2}\,dx =$$

$$= \frac{x^2}{2}\log(1+4x^2) - \int \frac{x+4x^3-x}{1+4x^2}\,dx =$$

$$= \frac{x^2}{2}\log(1+4x^2) - \int x\,dx + \frac{1}{8}\int \frac{8x}{1+4x^2}\,dx =$$

$$= \frac{x^2}{2}\log(1+4x^2) - \frac{x^2}{2} + \frac{1}{8}\log(1+4x^2) + C$$

$$\int \sqrt{x}\log x\,dx = \frac{2}{3}x^{3/2}\log x - \frac{2}{3}\int x^{3/2}\frac{1}{x}\,dx = \frac{2}{3}x^{3/2}\log x - \frac{2}{3}\int x^{\frac{1}{2}}\,dx = \frac{2}{3}x^{\frac{3}{2}}\log x - \frac{4}{9}x^{3/2} + C =$$

$$= \frac{2}{9}x^{3/2}(3\log x - 2) + C$$

$f'(x) = \sqrt{x}$ $\qquad f(x) = \frac{2}{3}x^{3/2}$

$g(x) = \log x$ $\qquad g'(x) = \frac{1}{x}$

$$\int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} - \int x\,\frac{1}{2}\,\frac{(-2x)}{\sqrt{1-x^2}}\,dx =$$

$$= x\sqrt{1-x^2} + \int \frac{x^2}{\sqrt{1-x^2}}\,dx =$$

$$= x\sqrt{1-x^2} - \int \frac{1-x^2-1}{\sqrt{1-x^2}}\,dx =$$

$$= x\sqrt{1-x^2} - \int \frac{1-x^2}{\sqrt{1-x^2}}\,dx + \int \frac{1}{\sqrt{1-x^2}}\,dx =$$

$$= x\sqrt{1-x^2} - \int \sqrt{1-x^2}\,dx + \arcsin x$$

$$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx = e^x \sin x - e^x \cos x - \int e^x \sin x\,dx$$

$$2\int e^x \sin x\,dx = e^x(\sin x - \cos x) \quad \rightarrow \quad \int e^x \sin x\,dx = \frac{1}{2} e^x (\sin x - \cos x) + C$$

$$\int x \operatorname{arctg} x \, dx = \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx = \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} \int \frac{x^2+1-1}{1+x^2} dx =$$

$$= \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} x + \frac{1}{2} \int \frac{1}{1+x^2} dx = \frac{x^2}{2} \operatorname{arctg} x - \frac{x}{2} + \frac{1}{2} \operatorname{arctg} x + C =$$

$$= -\frac{x}{2} + \frac{x^2+1}{2} \operatorname{arctg} x + C$$

$$\int e^x \log(3+e^x) \, dx = \int \log(3+t) \, dt = t \log(3+t) - \int \frac{t}{3+t} dt =$$

$e^x = t$

$e^x dx = dt$

$$= t \log(3+t) - \int \frac{3+t-3}{3+t} dt =$$

$$= t \log(3+t) - t + 3 \log(3+t) + C =$$

$$= (3+t) \log(3+t) - t + C =$$

$$= (3+e^x) \log(3+e^x) - e^x + C$$

$$\int \operatorname{arctg} \frac{1}{x+1}\, dx =$$

$$g(x) = \operatorname{arctg} \frac{1}{x+1}$$

$$g'(x) = \frac{1}{1+\frac{1}{(x+1)^2}} \cdot \left(-\frac{1}{(x+1)^2}\right) = -\frac{1}{x^2+2x+2}$$

$$= x \operatorname{arctg} \frac{1}{x+1} + \int \frac{x}{x^2+2x+2}\, dx =$$

$$= x \operatorname{arctg} \frac{1}{x+1} + \frac{1}{2} \int \frac{2x+2-2}{x^2+2x+2}\, dx =$$

$$= x \operatorname{arctg} \frac{1}{x+1} + \frac{1}{2} \int \frac{2x+2}{x^2+2x+2}\, dx - \int \frac{dx}{(x+1)^2+1} =$$

$$= x \operatorname{arctg} \frac{1}{x+1} + \frac{1}{2} \log\left(x^2+2x+2\right) - \operatorname{arctg}(x+1) + C$$

$$\int \operatorname{arctg}(2x+3)\, dx = \frac{1}{2} \int \operatorname{arctg} t\, dt = \frac{1}{2} t \operatorname{arctg} t - \frac{1}{2} \int \frac{t}{1+t^2}\, dt =$$

$$2x+3 = t$$

$$2dx = dt$$

$$= \frac{1}{2} t \operatorname{arctg} t - \frac{1}{4} \int \frac{2t}{1+t^2}\, dt =$$

$$= \frac{1}{2} t \operatorname{arctg} t - \frac{1}{4} \log\left(1+t^2\right) + C =$$

$$= \frac{1}{2}(2x+3) \operatorname{arctg}(2x+3) - \frac{1}{4} \log\left(4x^2+12x+10\right) + C$$

# Integrali definiti

# Calcolo integrali definiti

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(1 - \frac{1}{\sin^2 x}\right) dx = \Big[x + \operatorname{cotg} x\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{\pi}{3} + \operatorname{cotg}\frac{\pi}{3} - \frac{\pi}{4} - \operatorname{cotg}\frac{\pi}{4} = \frac{\pi}{12} + \frac{\sqrt{3}}{3} - 1 = \frac{\pi - 12 + 4\sqrt{3}}{12}$$

$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{2}{1+x^2}\, dx = \Big[2 \operatorname{arctg} x\Big]_{-\sqrt{3}}^{\sqrt{3}} = 2 \operatorname{arctg}\sqrt{3} - 2 \operatorname{arctg}(-\sqrt{3}) = 2 \cdot \frac{\pi}{3} - 2 \cdot \left(-\frac{\pi}{3}\right) = \frac{4\pi}{3}$$

$$\int_0^2 \frac{3x^2}{x^3+1}\, dx = \Big[\log|x^3+1|\Big]_0^2 = \log 9 - \log 1 = \log 9$$

$$\int_0^1 \frac{e^x dx}{e^x+2} = \Big[\log(e^x+2)\Big]_0^1 = \log(e+2) - \log 3 = \log\frac{2+e}{3}$$

$$\int_0^{\frac{\pi}{2}} \sin^2 x \cos x \, dx = \int_0^1 t^2 \, dt = \left[\frac{1}{3} t^3\right]_0^1 = \frac{1}{3}$$

$t = \sin x$

$dt = \cos x \, dx$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2 x \sin x \, dx = -\int_{\frac{\sqrt{2}}{2}}^{0} t^2 \, dt = \int_0^{\frac{\sqrt{2}}{2}} t^2 \, dt = \left[\frac{1}{3} t^3\right]_0^{\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{12}$$

$t = \cos x$

$dt = -\sin x \, dx$

$$\int_1^4 \frac{e^{2+\sqrt{x}}}{\sqrt{x}} \, dx = \int_1^2 2 e^{2+t} \, dt = \left[2 e^{2+t}\right]_1^2 = 2e^4 - 2e^3 = 2e^3(e-1)$$

$t = \sqrt{x}$

$dt = \frac{1}{2\sqrt{x}} \, dx$

$$\int_{-1}^{0} \frac{1}{x^2+5x+6}\,dx \qquad f(x) = \frac{1}{x^2+5x+6} = \frac{1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3} = \frac{(A+B)x + 3A+2B}{(x+2)(x+3)}$$

$$\begin{cases} A+B=0 \\ 3A+2B=1 \end{cases} \qquad \begin{cases} B=-A \\ A=1 \end{cases} \qquad \begin{cases} B=-1 \\ A=1 \end{cases} \qquad f(x) = \frac{1}{x+2} - \frac{1}{x+3}$$

$$\int_{-1}^{0} \left(\frac{1}{x+2} - \frac{1}{x+3}\right) dx = \Big[\log|x+2| - \log|x+3|\Big]_{-1}^{0} = \log 2 - \log 3 - \log 1 + \log 2 = 2\log 2 - \log 3$$

$$\int_{2}^{3} \frac{3x+2}{x^2-5x+4}\,dx \qquad f(x) = \frac{3x+2}{x^2-5x+4} = \frac{3x+2}{(x-1)(x-4)} = \frac{A}{x-1} + \frac{B}{x-4} = \frac{(A+B)x - 4A - B}{(x-1)(x-4)}$$

$$\begin{cases} A+B=3 \\ -4A-B=2 \end{cases} \qquad \begin{cases} A=3-B \\ -12+4B-B=2 \end{cases} \qquad \begin{cases} A=-\frac{5}{3} \\ B=\frac{14}{3} \end{cases}$$

$$\int_{2}^{3} \frac{3x+2}{x^2-5x+4}\,dx = \int_{2}^{3} \left(\frac{-\frac{5}{3}}{x-1} + \frac{\frac{14}{3}}{x-4}\right) dx = \left[-\frac{5}{3}\log|x-1| + \frac{14}{3}\log|x-4|\right]_{2}^{3} =$$

$$= -\frac{5}{3}\log 2 + \frac{14}{3}\log 1 + \frac{5}{3}\log 1 - \frac{14}{3}\log 2 = -\frac{19}{3}\log 2$$

$$\int x \sin x \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + C$$

$$\int_0^{\frac{\pi}{2}} x \sin x \, dx = \Big[ -x \cos x + \sin x \Big]_0^{\frac{\pi}{2}} = 1$$

$$\int \log x \, dx = x \log x - \int dx = x \log x - x + C$$

$$\int_2^3 \log x \, dx = \Big[ x(\log x - 1) \Big]_2^3 = 3 \log 3 - 3 - 2 \log 2 + 2 = \log 27 - \log 4 - 1 = \log \frac{27}{4} - 1$$

$$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C$$

$$\int_2^3 x e^x \, dx = \Big[ x e^x - e^x \Big]_2^3 = 3e^3 - e^3 - 2e^2 + e^2 = 2e^3 - e^2 = e^2 (2e - 1)$$

$$\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$$

$$\int e^x \cos x \, dx = \frac{1}{2} e^x \left( \cos x + \sin x \right) + C$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} e^x \cos x \, dx = \left[ \frac{1}{2} e^x \left( \cos x + \sin x \right) \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{\sqrt{3}+1}{4} \left( e^{\frac{\pi}{3}} - e^{\frac{\pi}{6}} \right)$$

$$\int \sqrt{1-x^2} \, dx = \int \cos^2 t \, dt = \int \frac{1+\cos 2t}{2} \, dt = \frac{1}{2} t + \frac{1}{4} \sin 2t + C = \frac{1}{2} \arcsin x + \frac{1}{2} x \sqrt{1-x^2} + C$$

$x = \sin t \qquad \sqrt{1-x^2} = \cos t$

$dx = \cos t \, dt$

$$\int_{\frac{1}{2}}^{1} \sqrt{1-x^2} \, dx = \left[ \frac{1}{2} \arcsin x + \frac{1}{2} x \sqrt{1-x^2} \right]_{\frac{1}{2}}^{1} = \frac{\pi}{4} - \frac{\pi}{12} - \frac{1}{4} \cdot \frac{\sqrt{3}}{2} = \frac{\pi}{6} - \frac{\sqrt{3}}{8}$$

# Calcolo aree figure piane

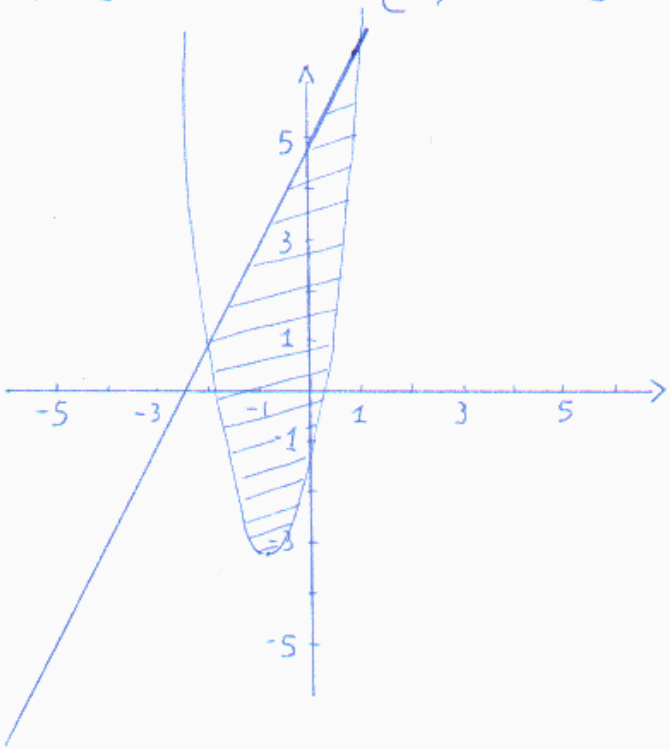
Calcolare l'area della regione di piano limitata dalla retta $y=2x+5$ e dalla parabola $y=3x^2+5x-$.

$$\begin{cases} y = 3x^2+5x-1 \\ y = 2x+5 \end{cases} \rightarrow \begin{cases} 2x+5 = 3x^2+5x-1 \\ y = 2x+5 \end{cases} \rightarrow \begin{cases} 3x^2+3x-6=0 \\ y = 2x+5 \end{cases} \rightarrow \begin{cases} x^2+x-2=0 \\ y = 2x+5 \end{cases}$$

$$x_{1,2} = \frac{-1 \pm 3}{2} = \begin{cases} -2 \\ 1 \end{cases}$$

$A(-2;1)$

$B(1;7)$




$$S = \int_{-2}^{1} \left[ (2x+5) - (3x^2+5x-1) \right] dx =$$

$$= \int_{-2}^{1} (-3x^2-3x+6)\, dx = \left[ -x^3 - \frac{3x^2}{2} + 6x \right]_{-2}^{1} =$$

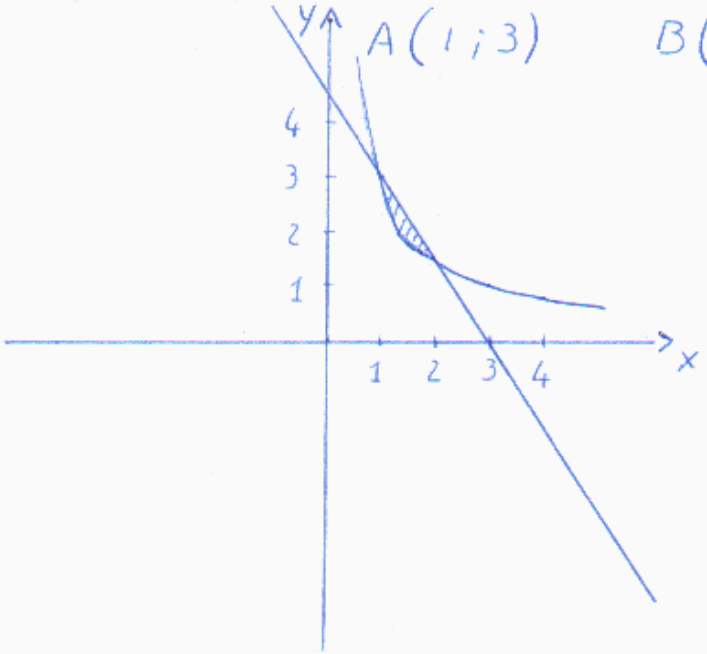
$$= -1 - \frac{3}{2} + 6 - 8 + 6 + 12 = \frac{27}{2}$$

Calcolare l'area della regione di piano limitata dalla retta 3x+2y-9=0 e dall'iperbole xy=3.

$$\begin{cases} y = \frac{3}{x} \\ 3x+2y-9=0 \end{cases} \rightarrow \begin{cases} y = \frac{3}{x} \\ 3x-9+\frac{6}{x}=0 \end{cases} \rightarrow \begin{cases} y = \frac{3}{x} \\ x^2-3x+2=0 \end{cases}$$

$$x_{1,2} = \frac{3 \pm 1}{2} = \begin{cases} 1 \\ 2 \end{cases}$$

$A(1;3)$ $\quad B\left(2;\frac{3}{2}\right)$




$$S = \int_1^2 \left[\left(-\frac{3}{2}x+\frac{9}{2}\right)-\frac{3}{x}\right]dx =$$

$$= \int_1^2 \left(-\frac{3}{2}x+\frac{9}{2}-\frac{3}{x}\right)dx =$$

$$= \left[-\frac{3}{4}x^2+\frac{9}{2}x-3\log|x|\right]_1^2 =$$

$$= -3+9-3\log 2+\frac{3}{4}-\frac{9}{2} = \frac{9}{4}-\log 8$$

Calcolare l'area della regione di piano limitata dall'asse x e dalla retta tangente alla parabola $y=-x^2+5x-4$ nel suo punto di ascissa $x=2$.

$y=-x^2+5x-4$

$V\left(\frac{5}{2};\frac{9}{4}\right)$

$A(1;0)$ $B(4;0)$

Tangente alla curva nel punto di ascissa $x=2$

$f'(x)=-2x+5$

$m=f'(2)=1$

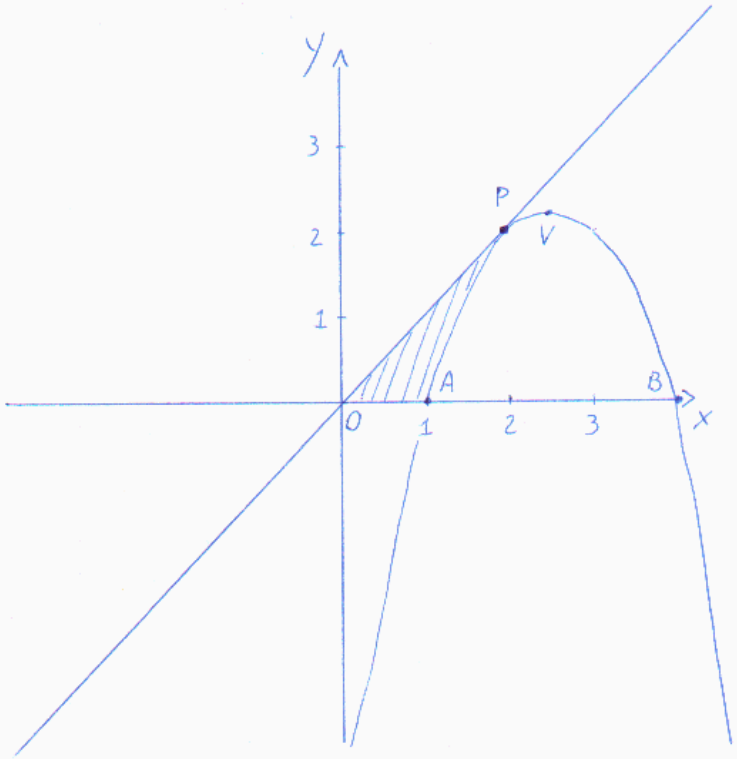
$x=2 \rightarrow y=-4+10-4=2$

$P(2;2)$

$y-y_P=m(x-x_P)$

$y-2=x-2$

$y=x$ retta tangente




$$S=\int_0^2 x\,dx-\int_1^2(-x^2+5x-4)\,dx=\left[\frac{x^2}{2}\right]_0^2-\left[-\frac{x^3}{3}+\frac{5x^2}{2}-4x\right]_1^2=$$

$$=2-\left(-\frac{8}{3}+10-8+\frac{1}{3}-\frac{5}{2}+4\right)=2-\frac{7}{6}=\frac{5}{6}$$

Calcolare l'area della regione di piano limitata dalle rette tangenti alla parabola $y=(-4/9)x^2+(8/3)x$ nei punti d'intersezione della stessa con la retta $y=x$.

$$\begin{cases} y = -\frac{4}{9}x^2 + \frac{8}{3}x \\ y = x \end{cases} \rightarrow \begin{cases} x = -\frac{4}{9}x^2 + \frac{8}{3}x \\ y = x \end{cases} \rightarrow \begin{cases} 9x = -4x^2 + 24x \\ y = x \end{cases} \rightarrow \begin{cases} 4x^2 - 15x = 0 \\ y = x \end{cases}$$

$A(0;0)$

$B\left(\frac{15}{4};\frac{15}{4}\right)$

Tangente in A:

$y' = -\frac{8}{9}x + \frac{8}{3}$

$m = f'(0) = \frac{8}{3}$

$y - y_A = m(x - x_A) \rightarrow y = \frac{8}{3}x$

Tangente in B:

$m = f'\left(\frac{15}{4}\right) = -\frac{\cancel{8}^{2}}{\cancel{9}_{3}} \cdot \frac{\cancel{15}^{5}}{\cancel{4}_{1}} + \frac{8}{3} = -\frac{10}{3} + \frac{8}{3} = -\frac{2}{3}$

$y - y_B = m(x - x_B) \rightarrow y - \frac{15}{4} = -\frac{2}{3}\left(x - \frac{15}{4}\right)$

$y = -\frac{2}{3}x + \frac{5}{2} + \frac{15}{4}$

$y = -\frac{2}{3}x + \frac{25}{4}$

C punto di intersezione delle rette tangenti in A e in B

$$\begin{cases} y = \frac{8}{3}x \\ y = -\frac{2}{3}x + \frac{25}{4} \end{cases} \rightarrow \begin{cases} y = \frac{8}{3}x \\ \frac{8}{3}x = -\frac{2}{3}x + \frac{25}{4} \end{cases} \rightarrow \begin{cases} y = \frac{8}{3}x \\ \frac{10}{3}x = \frac{25}{4} \end{cases} \rightarrow \begin{cases} y = \frac{8}{3} \cdot \frac{15}{8} = 5 \\ x = \frac{\cancel{25}^{5}}{4} \cdot \frac{3}{\cancel{10}_{2}} = \frac{15}{8} \end{cases} \qquad C\left(\frac{15}{8}; 5\right)$$

$$S = \int_0^{\frac{15}{8}} \frac{8}{3}x\,dx + \int_{\frac{15}{8}}^{\frac{15}{4}} \left(-\frac{2}{3}x + \frac{25}{4}\right)dx - \int_0^{\frac{15}{4}} \left(-\frac{4}{9}x^2 + \frac{8}{3}x\right)dx$$

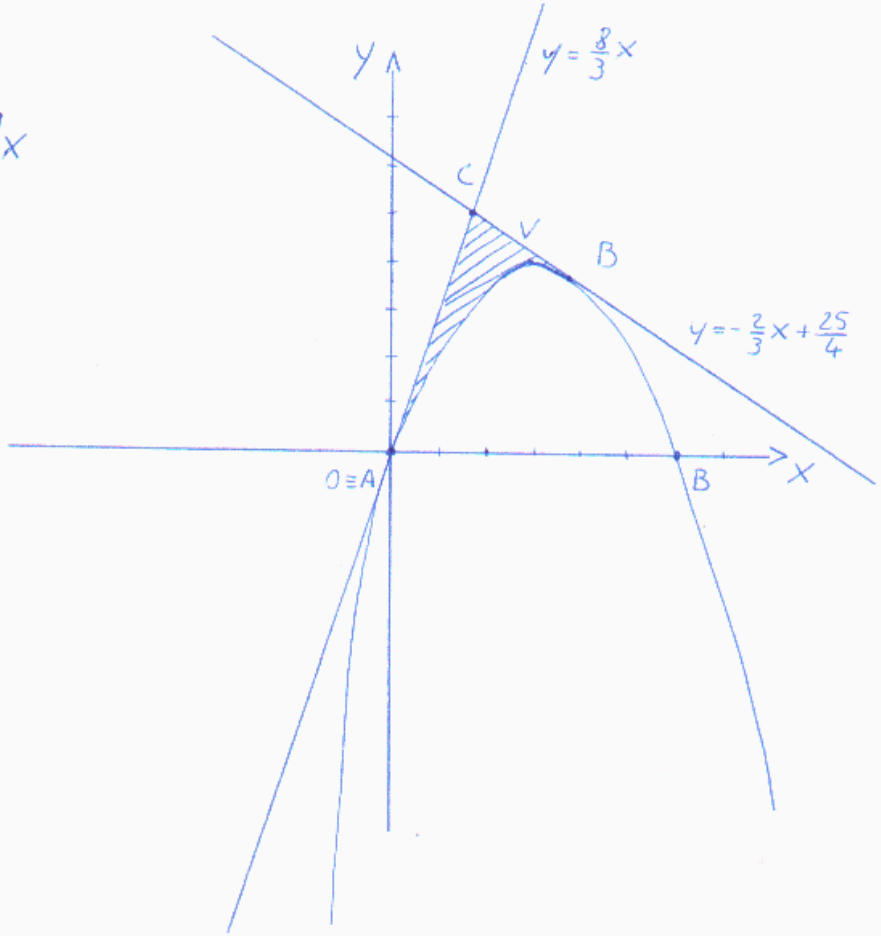
$$S = \left[\frac{4}{3}x^2\right]_0^{\frac{15}{8}} + \left[-\frac{x^2}{3} + \frac{25}{4}x\right]_{\frac{15}{8}}^{\frac{15}{4}} - \left[-\frac{4}{27}x^3 + \frac{4}{3}x^2\right]_0^{\frac{15}{4}}$$

$$S = \frac{4}{3} \cdot \frac{225}{64} - \frac{1}{3} \cdot \frac{225}{16} + \frac{25}{4} \cdot \frac{15}{4} + \frac{1}{3} \cdot \frac{225}{64} - \frac{25}{4} \cdot \frac{15}{8} +$$

$$+ \frac{4}{27} \cdot \frac{3375}{64} - \frac{4}{3} \cdot \frac{225}{16}$$

$$S = \cancel{\frac{75}{16}} - \cancel{\frac{75}{16}} + \frac{375}{16} + \frac{75}{64} - \frac{375}{32} + \frac{125}{16} - \frac{75}{4}$$

$$S = \frac{125}{64}$$



# Valor medio di una funzione

Calcolare il valor medio delle funzioni nell'intervallo indicato.

$y = x^2 - 4$

$[0;4]$

$$f(c) = \frac{\int_a^b f(x)\,dx}{b-a}$$

$$f(c) = \frac{1}{4}\int_0^4 (x^2-4)\,dx = \frac{1}{4}\left[\frac{x^3}{3} - 4x\right]_0^4 = \frac{1}{4}\left(\frac{64}{3} - 16\right) = \frac{1}{4}\cdot\frac{64-48}{3} = \frac{1}{4}\cdot\frac{16}{3} = \frac{4}{3}$$

$y = \frac{1}{x}$

$[1;e]$

$$f(c) = \frac{1}{e-1}\int_1^e \frac{1}{x}\,dx = \frac{1}{e-1}\left[\log|x|\right]_1^e = \frac{\log e - \log 1}{e-1} = \frac{1}{e-1}$$

$y = \cos x$

$\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$

$$f(c) = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx = \frac{1}{\pi}\left[\sin x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{\pi}\left(1-(-1)\right) = \frac{2}{\pi}$$

# Calcolo volume solidi di rotazione

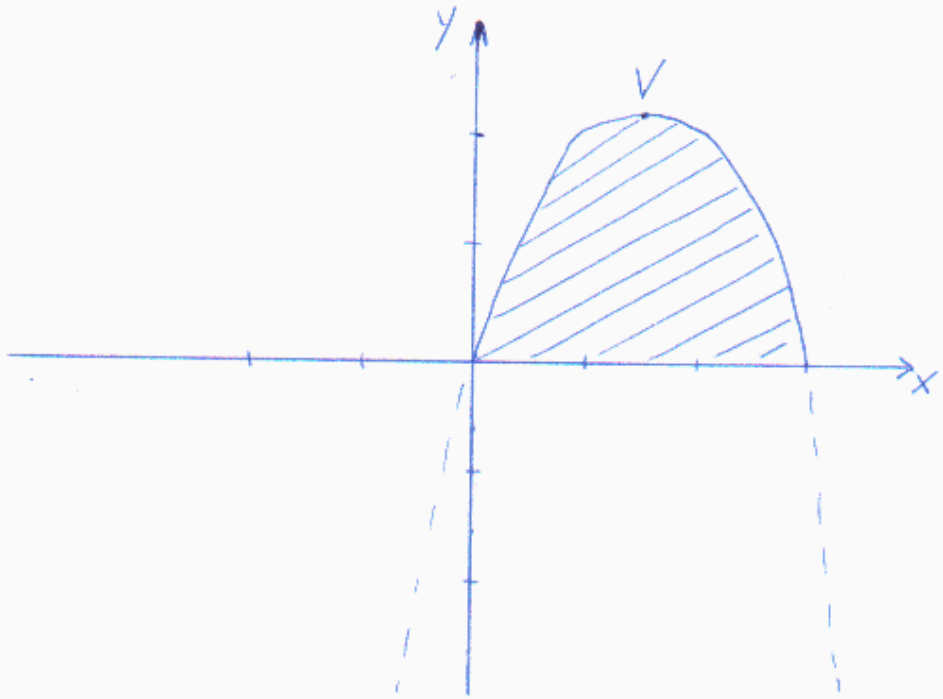
Calcolare il volume del solido ottenuto facendo ruotare intorno all'asse x di un giro completo la parabola $y=-x^2+3x$

$y=-x^2+3x$

$V\left(\frac{3}{2};\frac{9}{4}\right)$ vertice

$-x^2+3x=0$

$(0;0)$ $(3;0)$




$f(x)=-x^2+3x$

$$V=\pi\int_a^b f^2(x)\,dx$$

$$V=\pi\int_0^3\left(-x^2+3x\right)^2dx=\pi\int_0^3\left(x^4-6x^3+9x^2\right)dx=\pi\left[\frac{x^5}{5}-\frac{3}{2}x^4+3x^3\right]_0^3=$$

$$=\pi\left(\frac{243}{5}-\frac{243}{2}+81\right)=\frac{\pi}{10}\left(486-1215+810\right)=\frac{81\pi}{10}$$

Calcolare il volume del solido ottenuto facendo ruotare intorno all'asse x di un giro completo la parte di funzione $y=x^3-6x^2+9x$ che interseca l'asse delle x.

$$y = x^3 - 6x^2 + 9x$$

$$f(x) = 0$$

$$x(x^2 - 6x + 9) = 0$$

$$x(x-3)^2 = 0$$

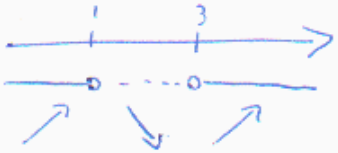
$(0;0)$ $(3;0)$

$$f'(x) = 3x^2 - 12x + 9$$

$$f'(x) = 0$$

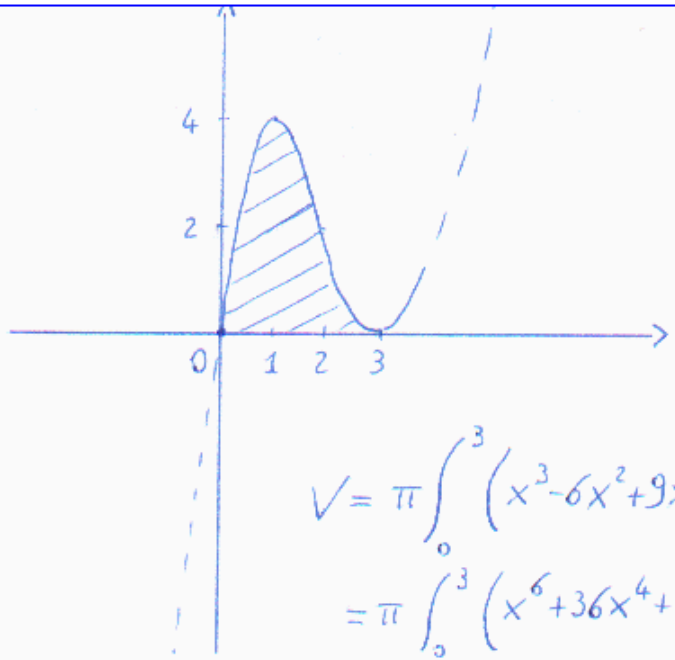
$$x^2 - 4x + 3 = 0$$

$$x_1 = 1 \quad \vee \quad x_2 = 3$$




$(1;4)$ max

$(3;0)$ min.




$$V = \pi \int_0^3 (x^3 - 6x^2 + 9x)^2 \, dx =$$

$$= \pi \int_0^3 (x^6 + 36x^4 + 81x^2 - 12x^5 + 18x^4 - 108x^3) \, dx =$$

$$= \pi \int_0^3 (x^6 - 12x^5 + 54x^4 - 108x^3 + 81x^2) \, dx =$$

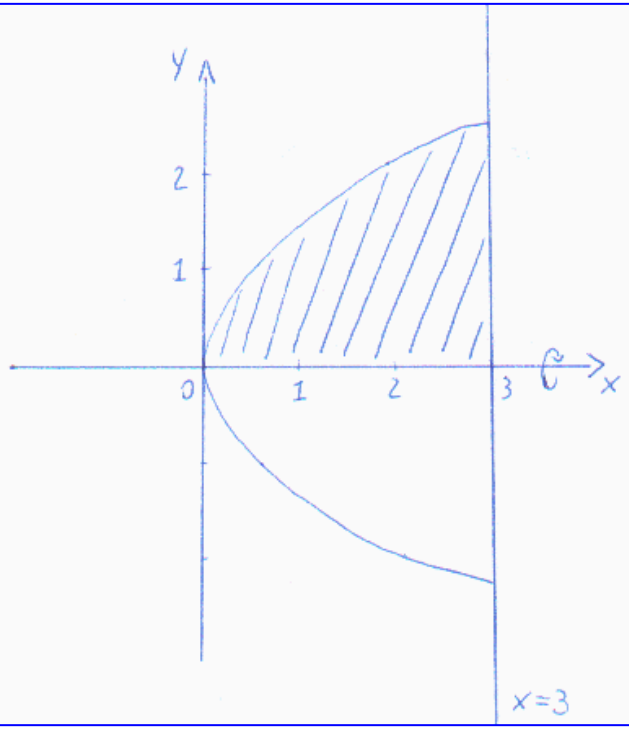
$$= \pi \left[ \frac{x^7}{7} - 2x^6 + \frac{54}{5}x^5 - 27x^4 + 27x^3 \right]_0^3 =$$

$$= \pi \left( \frac{2187}{7} - 1458 + \frac{13\,122}{5} - 2187 + 729 \right) = \frac{729\pi}{35}$$

Calcolare il volume del solido ottenuto facendo ruotare intorno all'asse x di un giro completo la parte di piano individuata dalla curva $y^2=2x$ e dalla retta $x=3$.

$y^2 = 2x$

$x = 3$




$x = \frac{1}{2} y^2$ parabola con asse di simmetria // all'asse x

$y = \sqrt{2x} \rightarrow f(x) = \sqrt{2x}$

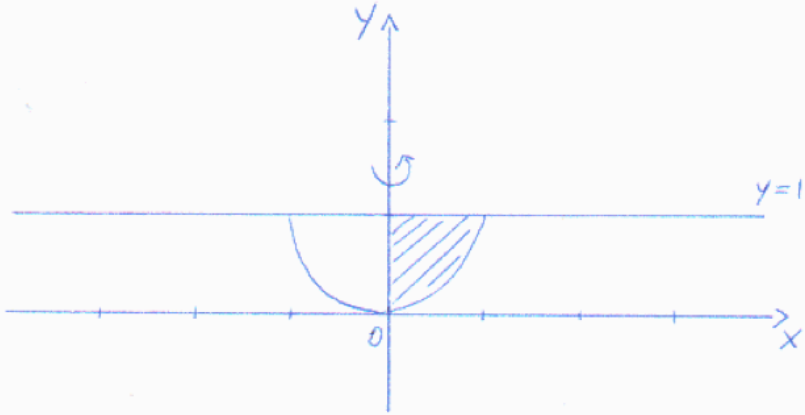
$$V = \pi \int_0^3 2x \, dx = \pi \left[ x^2 \right]_0^3 = 9\pi$$

Calcolare il volume del solido ottenuto facendo ruotare intorno all'asse y di un giro completo la parte di piano individuata dalla curva $y=x^2$ e dalla retta $y=1$.

$y = x^2$

$y = 1$

La rotazione è attorno all'asse y




$$V = \pi \int_{y_A}^{y_B} f^2(y)\, dy$$

$$y = x^2 \rightarrow x = \sqrt{y} \rightarrow f(y) = \sqrt{y}$$

$$V = \pi \int_0^1 y\, dy = \pi \left[ \frac{1}{2} y^2 \right]_0^1 = \frac{\pi}{2}$$

Calcolare il volume del solido ottenuto facendo ruotare intorno all'asse x di un giro completo la parte superiore dell'ellisse $(x^2/9)+(y^2/4)=1$.

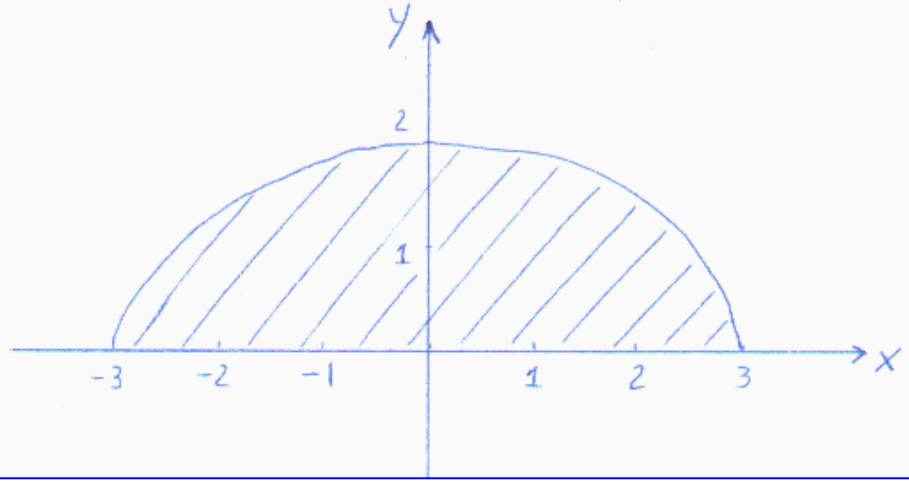
$\frac{x^2}{9}+\frac{y^2}{4}=1$ ellisse

ruota attorno all'asse x

$$\frac{y^2}{4}=1-\frac{x^2}{9} \rightarrow y^2=4\left(1-\frac{x^2}{9}\right) \rightarrow f^2(x)=4-\frac{4}{9}x^2$$

$$V=\pi\int_{-3}^{+3}\left(4-\frac{4}{9}x^2\right)dx=2\pi\int_0^3\left(4-\frac{4}{9}x^2\right)dx=$$

$$=2\pi\left[4x-\frac{4}{27}x^3\right]_0^3=2\pi(12-4)=16\pi$$

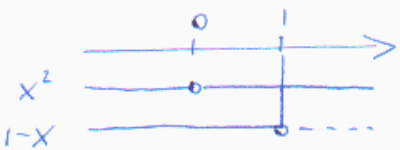


Calcolare il volume del solido ottenuto facendo ruotare intorno all'asse x di un giro completo la parte di funzione $y=-x^3+x^2$ che interseca l'asse delle x.

$y = x^2 - x^3 = x^2(1-x)$

$D = \mathbb{R}$

$(0;0) \quad (1;0)$




$I.P. = (-\infty;0) \cup (0;1)$

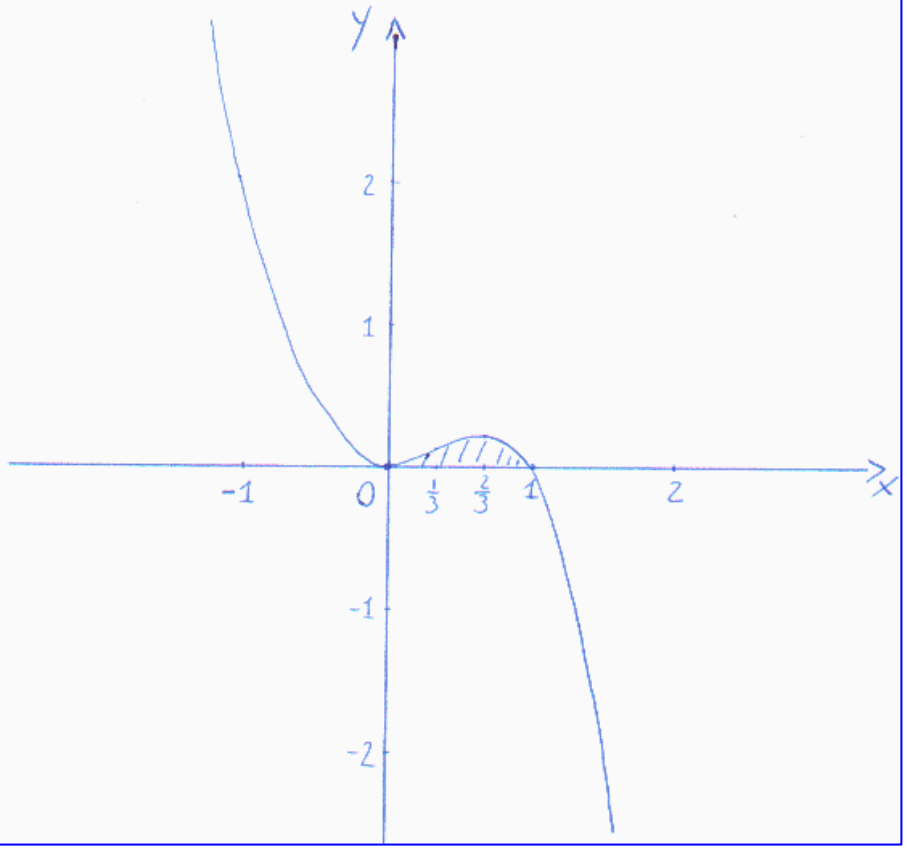
$I.N. = (1;+\infty)$

$\lim_{x\to-\infty} f(x) = +\infty$

$\lim_{x\to+\infty} f(x) = -\infty$

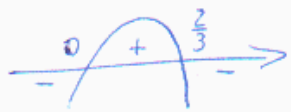
$\lim_{x\to\infty} \frac{f(x)}{x} = \infty$

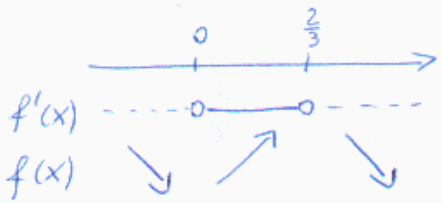
Non esistono asintoti orizzontali e obliqui



$y' = 2x - 3x^2$

$f'(x) > 0$



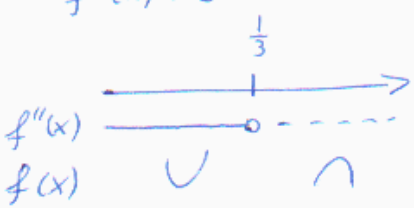




$(0;0)$ min

$\left(\frac{2}{3}; \frac{4}{27}\right)$ max

$f''(x) = 2 - 6x$

$f''(x) > 0$




$\left(\frac{1}{3}; \frac{2}{27}\right)$ punto di flesso discendente

(1) $A = \int_0^1 f(x)\,dx = \int_0^1 (x^2 - x^3)\,dx =$

$= \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{4-3}{12} = \frac{1}{12}$

(2) $V = \pi \int_0^1 f^2(x)\,dx = \pi \int_0^1 (x^2 - x^3)^2\,dx =$

$= \pi \int_0^1 (x^4 - 2x^5 + x^6)\,dx =$

$= \pi \left[\frac{x^5}{5} - \frac{1}{3}x^6 + \frac{x^7}{7}\right]_0^1 =$

$= \pi \left(\frac{1}{5} - \frac{1}{3} + \frac{1}{7}\right) =$

$= \pi \, \frac{21 - 35 + 15}{105} = \frac{\pi}{105}$

# Integrali impropri

$$\int_1^{+\infty} \frac{dx}{x\sqrt{x}} = \lim_{b\to+\infty} \int_1^b x^{-\frac{3}{2}}dx = \lim_{b\to+\infty} \left[-\frac{2}{\sqrt{x}}\right]_1^b = \lim_{b\to+\infty} \left(-\frac{2}{\sqrt{b}} + 2\right) = 2$$

$$\int_2^{+\infty} \frac{\sqrt{x}}{x\sqrt[3]{x^2}}dx = \lim_{b\to+\infty} \int_2^b \frac{x^{\frac{1}{2}}}{x\cdot x^{\frac{2}{3}}}dx = \lim_{b\to+\infty} \int_2^b \frac{x^{\frac{1}{2}}}{x^{5/3}}dx = \lim_{b\to+\infty} \int_2^b x^{-\frac{7}{6}}dx =$$

$$= \lim_{b\to+\infty} \left[-\frac{6}{\sqrt[6]{x}}\right]_2^b = \lim_{b\to+\infty} \left(-\frac{6}{\sqrt[6]{b}} + \frac{6}{\sqrt[6]{2}}\right) = \frac{6}{\sqrt[6]{2}}$$

$$\int_{-\infty}^0 e^{3x}dx = \lim_{a\to-\infty} \int_a^0 e^{3x}dx = \lim_{a\to-\infty} \left[\frac{1}{3}e^{3x}\right]_a^0 = \lim_{a\to-\infty} \left(\frac{1}{3} - \frac{1}{3}e^{3a}\right) = \frac{1}{3}$$

$$\int_{-\infty}^0 \frac{e^x}{1+e^x}dx = \lim_{a\to-\infty} \int_a^0 \frac{e^x}{1+e^x}dx = \lim_{a\to-\infty} \left[\log(1+e^x)\right]_a^0 =$$

$$= \lim_{a\to-\infty} \left(\log 2 - \log(1+e^a)\right) = \log 2 - \log 1 = \log 2$$

$$\int_{-\infty}^{+\infty} \frac{e^x}{1+e^{2x}}\,dx = \lim_{\substack{a\to-\infty \\ b\to+\infty}} \int_a^b \frac{e^x}{1+e^{2x}}\,dx = \lim_{\substack{a\to-\infty \\ b\to+\infty}} \Big[\operatorname{arctg} e^x\Big]_a^b =$$

$$= \lim_{\substack{a\to-\infty \\ b\to+\infty}} \left(\operatorname{arctg} e^b - \operatorname{arctg} e^a\right) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\int_1^{+\infty} \frac{e^{\frac{1}{x}}}{x^2}\,dx = \lim_{b\to+\infty} \int_1^b \frac{e^{\frac{1}{x}}}{x^2}\,dx = \lim_{b\to+\infty} \Big[-e^{\frac{1}{x}}\Big]_1^b =$$

$$= \lim_{b\to+\infty} \left(-e^{\frac{1}{x}} + e\right) = -e^0 + e = e - 1$$

$$\int_3^{+\infty} \frac{x\,dx}{x^2+1} = \lim_{b\to+\infty} \int_3^b \frac{x\,dx}{x^2+1} = \lim_{b\to+\infty} \left[\frac{1}{2}\log(x^2+1)\right]_3^b =$$

$$= \lim_{b\to+\infty} \left(\frac{1}{2}\log(b^2+1) - \frac{1}{2}\log 10\right) = +\infty$$

$$\int_{-\infty}^{-2} \frac{dx}{4x^2-9} = \lim_{a\to-\infty} \int_a^{-2} \frac{dx}{(2x+3)(2x-3)} =$$

$$= \lim_{a\to-\infty} \int_a^{-2} \left(-\frac{1}{6}\,\frac{1}{2x+3} + \frac{1}{6}\,\frac{1}{2x-3}\right) dx =$$

$$= \lim_{a\to-\infty} \left[\frac{1}{12}\log\left|\frac{2x-3}{2x+3}\right|\right]_a^{-2} =$$

$$= \lim_{a\to-\infty} \left(\frac{1}{12}\log 7 - \frac{1}{12}\log\left|\frac{2a-3}{2a+3}\right|\right) = \frac{1}{12}\log 7$$

$$f(x) = \frac{1}{(2x+3)(2x-3)} = \frac{A}{2x+3} + \frac{B}{2x-3}$$

$$f(x) = \frac{(2A+2B)x + 3B - 3A}{4x^2-9}$$

$$\begin{cases} 2A+2B = 0 \\ 3B-3A = 1 \end{cases} \quad \begin{cases} A = -B \\ 6B = 1 \end{cases} \quad \begin{cases} A = -\frac{1}{6} \\ B = \frac{1}{6} \end{cases}$$

$$\int_1^{+\infty} \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \lim_{b\to+\infty} \int_1^{b} \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \lim_{b\to+\infty} \Big[2\sin\sqrt{x}\Big]_1^b =$$

$$= \lim_{b\to+\infty} \left(2\sin\sqrt{b} - 2\sin 1\right) \quad \text{NON ESISTE}$$

L'integrale improprio $\displaystyle\int_1^{+\infty} \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

è indeterminato.

$$\int_{-\infty}^{+\infty} \frac{e^{\operatorname{arctg} x}}{1+x^2}\,dx = \lim_{\substack{a\to-\infty\\ b\to+\infty}} \int_a^b \frac{e^{\operatorname{arctg} x}}{1+x^2}\,dx = \lim_{\substack{a\to-\infty\\ b\to+\infty}} \Big[e^{\operatorname{arctg} x}\Big]_a^b =$$

$$= \lim_{\substack{a\to-\infty\\ b\to+\infty}} \left(e^{\operatorname{arctg} b} - e^{\operatorname{arctg} a}\right) = e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}} = \frac{e^{\pi}-1}{e^{\frac{\pi}{2}}}$$

$$\int_2^{+\infty} \frac{x}{(x-1)^4}\,dx = \lim_{b\to+\infty} \int_2^{b} \frac{x}{(x-1)^4}\,dx = \qquad t = x-1$$

$$= \lim_{b\to+\infty} \int_1^{b-1} \frac{t+1}{t^4}\,dt = \lim_{b\to+\infty} \int_1^{b-1} \left(\frac{1}{t^3} + \frac{1}{t^4}\right) dx =$$

$$= \lim_{b\to+\infty} \left[-\frac{1}{2t^2} - \frac{1}{3t^3}\right]_1^{b-1} =$$

$$= \lim_{b\to+\infty} \left(-\frac{1}{2(b-1)^2} - \frac{1}{3(b-1)^3} + \frac{1}{2} + \frac{1}{3}\right) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

$$\int_0^2 \frac{dx}{\sqrt[3]{x-2}} = \lim_{\epsilon\to0^+} \int_0^{2-\epsilon} \frac{dx}{\sqrt[3]{x-2}} = \lim_{\epsilon\to0^+} \left[\frac{3}{2}\sqrt[3]{(x-2)^2}\right]_0^{2-\epsilon} =$$

$$= \lim_{\epsilon\to0^+} \left(\frac{3}{2}\sqrt[3]{\epsilon^2} - \frac{3}{2}\sqrt[3]{4}\right) = -\frac{3}{2}\sqrt[3]{4}$$

$$\int_0^{\frac{3}{4}} \frac{dx}{\sqrt{3-4x}} = \lim_{\epsilon \to 0^+} \int_0^{\frac{3}{4}-\epsilon} \frac{dx}{\sqrt{3-4x}} = \lim_{\epsilon \to 0^+} \left[ -\frac{1}{2}\sqrt{3-4x} \right]_0^{\frac{3}{4}-\epsilon} =$$

$$= \lim_{\epsilon \to 0^+} \left( -\frac{1}{2}\sqrt{4\epsilon} + \frac{1}{2}\sqrt{3} \right) = \frac{\sqrt{3}}{2}$$

$$\int_2^3 \frac{dx}{\sqrt[3]{x-2}} = \lim_{\delta \to 0^+} \int_{2+\delta}^3 \frac{dx}{\sqrt[3]{x-2}} = \lim_{\delta \to 0^+} \left[ \frac{3}{2}\sqrt[3]{(x-2)^2} \right]_{2+\delta}^3 =$$

$$= \lim_{\delta \to 0^+} \left( \frac{3}{2} - \frac{3}{2}\sqrt[3]{\delta^2} \right) = \frac{3}{2}$$

$$\int_0^1 x^2 \log x \, dx = \lim_{\delta \to 0^+} \int_\delta^1 x^2 \log x \, dx =$$

$$= \lim_{\delta \to 0^+} \left( \left[ \frac{x^3}{3} \log x \right]_\delta^1 - \int_\delta^1 \frac{x^3}{3} \cdot \frac{1}{x} \, dx \right) =$$

$$= \lim_{\delta \to 0^+} \left[ \frac{x^3}{3} \log x - \frac{x^3}{9} \right]_\delta^1 =$$

$$= \lim_{\delta \to 0^+} \left( -\frac{1}{9} - \frac{\delta^3}{3} \log \delta + \frac{\delta^3}{9} \right) = -\frac{1}{9}$$

$$\int_1^4 \frac{dx}{x-1} = \lim_{\delta \to 0^+} \int_{1+\delta}^4 \frac{dx}{x-1} = \lim_{\delta \to 0^+} \Big[ \log|x-1| \Big]_{1+\delta}^4 =$$

$$= \lim_{\delta \to 0^+} \left( \log 3 - \log \delta \right) = +\infty$$

$$\int_{-1}^1 \frac{\arcsin^3 x}{\sqrt{1-x^2}}\, dx = \lim_{\substack{\epsilon \to 0^+ \\ \delta \to 0^+}} \int_{-1+\delta}^{1-\epsilon} \frac{\arcsin^3 x}{\sqrt{1-x^2}}\, dx = \lim_{\substack{\epsilon \to 0^+ \\ \delta \to 0^+}} \left[ \frac{1}{4} \arcsin^4 x \right]_{-1+\delta}^{1-\epsilon} =$$

$$= \lim_{\substack{\epsilon \to 0^+ \\ \delta \to 0^+}} \left( \frac{1}{4} \arcsin^4 (1-\epsilon) - \frac{1}{4} \arcsin^4 (-1+\delta) \right) =$$

$$= \frac{1}{4} \left( \frac{\pi}{2} \right)^4 - \frac{1}{4} \left( -\frac{\pi}{2} \right)^4 = 0$$

$$\int_0^1 \frac{e^{1-\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{\delta\to 0^+} \int_\delta^1 \frac{e^{1-\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{\delta\to 0^+} \left[-2e^{1-\sqrt{x}}\right]_\delta^1 =$$

$$= \lim_{\delta\to 0^+} \left(-2 + 2e^{1-\sqrt{\delta}}\right) = 2e - 2 = 2(e-1)$$

$$\int_{\frac{\sqrt{2}}{2}}^1 \frac{\arcsin x}{\sqrt{1-x^2}}\,dx = \lim_{\epsilon\to 0^+} \int_{\frac{\sqrt{2}}{2}}^{1-\epsilon} \frac{\arcsin x}{\sqrt{1-x^2}}\,dx = \lim_{\epsilon\to 0^+} \left[\frac{1}{2}\arcsin^2 x\right]_{\frac{\sqrt{2}}{2}}^{1-\epsilon} =$$

$$= \lim_{\epsilon\to 0^+} \left(\frac{1}{2}\arcsin^2(1-\epsilon) - \frac{1}{2}\arcsin^2\frac{\sqrt{2}}{2}\right) =$$

$$= \frac{1}{2}\left(\frac{\pi^2}{4} - \frac{\pi^2}{16}\right) = \frac{3\pi^2}{32}$$

# Equazioni differenziali

# Equazioni differenziali a variabili separabili

$$y' = \boxed{g(x)} \cdot \boxed{h(y)}$$

funzione di x — funzione di y

$$y y' = 3$$

$$y\,dy = 3\,dx$$

$$\int y\,dy = \int 3\,dx$$

$$\frac{1}{2} y^2 = 3x + c_1$$

$$y^2 = 6x + 2c_1$$

$$y^2 = 6x + c$$

$$y' + 2xy^2 = 0$$

$$\frac{dy}{dx} = -2xy^2 \rightarrow \frac{dy}{y^2} = -2x\,dx$$

$$\int \frac{dy}{y^2} = -\int 2x\,dx$$

$$-\frac{1}{y} = -x^2 + c_1$$

$$y = \frac{1}{x^2 - c_1}$$

$$c = -c_1$$

$$y = \frac{1}{x^2 + c}$$

$$y' = xy$$

$$\frac{dy}{dx} = xy$$

$$\frac{dy}{y} = x\,dx$$

$$\int \frac{dy}{y} = \int x\,dx$$

$$\log|y| = \frac{x^2}{2} + c_1$$

$$|y| = e^{\frac{x^2}{2}+c_1} \rightarrow |y| = e^{c_1} \cdot e^{\frac{x^2}{2}}$$

$$y = \pm e^{c_1} \cdot e^{\frac{x^2}{2}}$$

$$c = \pm e^{c_1} \qquad c_1 \in \mathbb{R}, \quad c \neq 0$$

$$y = c e^{\frac{1}{2}x^2} \qquad c \neq 0$$

$y = 0$ è soluzione di $y' = xy$

$$y = c e^{\frac{1}{2}x^2} \qquad c \in \mathbb{R}$$

$$y' = 2y + 3$$

$$\frac{dy}{2y+3} = dx$$

$$\int \frac{dy}{2y+3} = \int dx$$

$$\frac{1}{2}\log|2y+3| = x + c_1$$

$$\log|2y+3| = 2x + 2c_1$$

$$|2y+3| = e^{2c_1} \cdot e^{2x}$$

$$2y+3 = \pm e^{2c_1} \cdot e^{2x}$$

$$c = \pm e^{2c_1}$$

$$2y = c e^{2x} - 3$$

$$y = \frac{c e^{2x} - 3}{2} \qquad c \neq 0$$

$y = -\frac{3}{2}$ è soluzione di $y' = 2y + 3$

$$y = \frac{c e^{2x} - 3}{2} \qquad c \in \mathbb{R}$$

$$y' - 3x^2y^2 = 0$$

$$\frac{dy}{dx} = 3x^2y^2$$

$$\frac{dy}{y^2} = 3x^2dx$$

$$\int \frac{dy}{y^2} = \int 3x^2dx$$

$$-\frac{1}{y} = x^3 + c_1$$

$$y = \frac{-1}{x^3 + c_1}$$

$$y' = \frac{2xy}{x^2-1}$$

$$\int \frac{dy}{y} = \int \frac{2x}{(x+1)(x-1)}\,dx$$

$$f(x) = \frac{2x}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{(A+B)x - A + B}{x^2-1}$$

$$\begin{cases} A+B=2 \\ -A+B=0 \end{cases} \qquad \begin{cases} 2A=2 \\ B=A \end{cases} \qquad \begin{cases} A=1 \\ B=1 \end{cases}$$

$$\int \frac{dy}{y} = \int \left( \frac{1}{x+1} + \frac{1}{x-1} \right) dx$$

$$\log|y| = \log|x^2-1| + c_1$$

$$|y| = e^{c_1} e^{\log|x^2-1|}$$

$$y = \pm e^{c_1}(x^2-1) \qquad c = \pm e^{c_1}$$

$$y + x^2 y' = xy$$

$$x^2 \frac{dy}{dx} = y(x-1)$$

$$\frac{dy}{y} = \frac{x-1}{x^2}\, dx$$

$$\int \frac{dy}{y} = \int \left( \frac{1}{x} - \frac{1}{x^2} \right) dx$$

$$\log |y| = \log |x| + \frac{1}{x} + c_1$$

$$|y| = e^{\frac{1}{x}} \cdot e^{\log |x|} \cdot e^{c_1}$$

$$y = \pm e^{c_1} x \cdot e^{\frac{1}{x}} \qquad K = \pm e^{c_1}$$

$$y = K x e^{\frac{1}{x}} \qquad K \in \mathbb{R}$$

# Equazioni differenziali lineari

$$\boxed{y'} + a(x) \cdot \boxed{y} = b(x)$$

derivata prima — funzione $y = f(x)$

**I grado**

$y' = 2xy - 2x^3$

$a(x) = 2x \rightarrow A(x) = \int 2x\,dx = x^2$

$b(x) = -2x^3$

$K(x) = \int (-2x^3)\, e^{-x^2} dx \qquad t = x^2 \qquad dt = 2x\,dx$

$K(x) = -\int t e^{-t} dt = t e^{-t} - \int e^{-t} dt = t e^{-t} + e^{-t} + c$

$K(x) = (x^2+1)\, e^{-x^2} + c$

$y = e^{x^2}\left[(x^2+1)\, e^{-x^2} + c\right] = x^2 + 1 + c e^{x^2}$

$$y' - 2y + e^x = 0$$

$$y' = 2y - e^x$$

$$a(x) = 2 \quad \rightarrow \quad A(x) = \int 2\,dx = 2x$$

$$b(x) = -e^x$$

$$K(x) = \int (-e^x)\, e^{-2x}\, dx = -\int e^{-x}\, dx = e^{-x} + c$$

$$y = e^{2x}\left(e^{-x} + c\right) = e^x\left(1 + c e^x\right)$$

$$y' - y = x^2$$

$$y' = y + x^2$$

$$a(x) = 1 \quad \rightarrow \quad A(x) = x$$

$$b(x) = x^2$$

$$K(x) = \int x^2 e^{-x}\, dx = -x^2 e^{-x} + \int 2x e^{-x}\, dx =$$

$$= -x^2 e^{-x} - 2x e^{-x} + 2\int e^{-x}\, dx =$$

$$= \left(-x^2 - 2x - 2\right) e^{-x} + c$$

$$y = e^x\left[\left(-x^2 - 2x - 2\right) e^{-x} + c\right]$$

$$y = -x^2 - 2x - 2 + c e^x$$

$$y' + xy + x = 0$$

$$y' = -xy - x$$

$$a(x) = -x \quad \rightarrow \quad A(x) = -\int x\,dx = -\frac{1}{2}x^2$$

$$b(x) = -x$$

$$K(x) = -\int x e^{\frac{1}{2}x^2} dx \qquad t = x^2 \qquad dt = 2x\,dx$$

$$K(x) = -\frac{1}{2}\int e^{\frac{1}{2}t} dt = -e^{\frac{1}{2}t} + C = -e^{\frac{1}{2}x^2} + C$$

$$y = e^{-\frac{1}{2}x^2}\left[-e^{\frac{1}{2}x^2} + C\right] = -1 + Ce^{-\frac{1}{2}x^2}$$

$$y' + y = \sin x$$

$$y' = -y + \sin x$$

$$a(x) = -1 \quad \rightarrow \quad A(x) = -x$$

$$b(x) = \sin x$$

$$K(x) = \int \sin x\, e^x dx = \sin x\, e^x - \int \cos x\, e^x dx$$

$$K(x) = (\sin x - \cos x)e^x - \int \sin x\, e^x dx$$

$$2\int \sin x\, e^x dx = (\sin x - \cos x)e^x$$

$$\int \sin x\, e^x dx = \frac{1}{2}e^x(\sin x - \cos x) + C$$

$$y = e^{-x}\left[\frac{1}{2}e^x(\sin x - \cos x) + C\right] = \frac{1}{2}(\sin x - \cos x) + Ce^{-x}$$

$$y' = \frac{2}{x} y + \frac{1}{x-1}$$

$$a(x) = \frac{2}{x} \longrightarrow A(x) = \int \frac{2}{x} dx = 2 \log|x| = \log x^2$$

$$b(x) = \frac{1}{x-1}$$

$$K(x) = \int \frac{1}{x-1} e^{-\log x^2} dx = \int \frac{1}{x^2(x-1)} dx$$

$$\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} = \frac{Ax^2 - Ax + Bx - B + Cx^2}{x^2(x-1)}$$

$$\begin{cases} A + C = 0 \\ -A + B = 0 \\ -B = 1 \end{cases} \rightarrow \begin{cases} A = -1 \\ B = -1 \\ C = 1 \end{cases}$$

$$K(x) = \int \left( -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1} \right) dx = -\log|x| + \frac{1}{x} + \log|x-1| + c$$

$$K(x) = \frac{1}{x} + \log\left|\frac{x-1}{x}\right| + c$$

$$y = x^2 \left[ \frac{1}{x} + \log\left|\frac{x-1}{x}\right| + c \right] = x \left( 1 + x \log\left|\frac{x-1}{x}\right| + cx \right)$$

$$y' - y = e^x$$

$$y' = y + e^x$$

$$a(x) = 1 \longrightarrow A(x) = x$$

$$b(x) = e^x$$

$$K(x) = \int e^x e^{-x} dx = \int dx = x + c$$

$$y = e^x (x + c)$$

$$y' - e^{2x} = y$$

$$y' = y + e^{2x}$$

$$a(x) = 1 \longrightarrow A(x) = x$$

$$b(x) = e^{2x}$$

$$K(x) = \int e^{2x} e^{-x} dx = \int e^x dx = e^x + c$$

$$y = e^x (e^x + c) = e^{2x} + c e^x$$

$$y' = \frac{1}{x} y - \frac{\log x}{x}$$

$x > 0$

$$a(x) = \frac{1}{x} \rightarrow A(x) = \int \frac{1}{x}\,dx = \log x$$

$$b(x) = -\frac{1}{x} \log x$$

$$K(x) = -\int \frac{1}{x} \log x \, e^{-\log x}\,dx = -\int \frac{1}{x^2} \log x \,dx$$

$$K(x) = \frac{1}{x} \log x - \int \frac{1}{x^2}\,dx = \frac{1}{x} \log x + \frac{1}{x} + c$$

$$y = x\left(\frac{1}{x} \log x + \frac{1}{x} + c\right) = \log x + 1 + cx$$

$$y' = \left(1 + \frac{2}{x}\right) y - e^x$$

$$a(x) = 1 + \frac{2}{x} \rightarrow A(x) = x + 2 \log|x| = x + \log x^2$$

$$b(x) = -e^x$$

$$K(x) = -\int e^x e^{-x - \log x^2}\,dx = -\int \frac{1}{x^2}\,dx = \frac{1}{x} + c$$

$$y = x^2 e^x \left(\frac{1}{x} + c\right) = e^x \left(x + cx^2\right)$$

$$(y'+y)x^2 = e^{-x}$$

$$y' = -y + \frac{e^{-x}}{x^2}$$

$$a(x) = -1 \rightarrow A(x) = -x$$

$$b(x) = \frac{e^{-x}}{x^2}$$

$$K(x) = \int \frac{e^{-x}}{x^2} e^{x} dx = \int \frac{dx}{x^2} = -\frac{1}{x} + c$$

$$y = e^{-x}\left(-\frac{1}{x} + c\right)$$

$$y' + 2xy = e^{-x^2}$$

$$y' = -2xy + e^{-x^2}$$

$$a(x) = -2x \rightarrow A(x) = -x^2$$

$$b(x) = e^{-x^2}$$

$$K(x) = \int e^{-x^2} e^{x^2} dx = \int dx = x + c$$

$$y = e^{-x^2}(x+c)$$

# Equazioni differenziali del 2° ordine non omogenee a coefficienti costanti

$$y'' + by' + cy = r(x)$$

$y'' - 4y' - 5y = 10$

$K^2 - 4K - 5 = 0$

$K_{1,2} = 2 \pm \sqrt{9} = \begin{cases} -1 \\ 5 \end{cases}$

$Y(x, c_1, c_2) = c_1 e^{-x} + c_2 e^{5x}$

integrale generale dell'eq. differenziale omogenea associata

$p(x) = 10$

$q(x) = a \qquad q'(x) = 0 \qquad q''(x) = 0$

$-5a = 10 \rightarrow a = -2$

$q(x) = -2$ integrale particolare dell'eq. differenziale

soluzione:

$$y = c_1 e^{-x} + c_2 e^{5x} - 2$$

$$y'' - 4y' + 4y = 8$$

$$K^2 - 4K + 4 = 0$$

$$(K-2)^2 = 0$$

$$K_1 = K_2 = 2$$

$$Y(x, c_1, c_2) = c_1 e^{2x} + c_2 x e^{2x}$$

$$p(x) = 8$$

$$q(x) = a \qquad q'(x) = q''(x) = 0$$

$$4a = 8 \rightarrow a = 2$$

soluzione:

$$y = (c_1 + c_2 x) e^{2x} + 2$$

$$y'' - y = 6$$

$$K^2 - 1 = 0$$

$$K = \pm 1$$

$$Y(x, c_1, c_2) = c_1 e^{-x} + c_2 e^{x}$$

$$p(x) = 6$$

$$q(x) = a \qquad q'(x) = q''(x) = 0$$

$$-a = 6 \rightarrow a = -6$$

soluzione: $y = c_1 e^{-x} + c_2 e^{x} - 6$

$y'' - y' = 3$

$K^2 - K = 0$

$K(K-1) = 0$

$K_1 = 0 \quad K_2 = 1$

$Y(x, c_1, c_2) = c_1 + c_2 e^x$

$p(x) = 3$

$q(x) = ax + b \qquad q'(x) = a \qquad q''(x) = 0$

$-a = 3 \rightarrow a = -3 \qquad q(x) = -3x + c_3$

soluzione: $y = c_2 e^x - 3x + c_4 \qquad c_4 = c_1 + c_3$

$y'' + y = 4$

$K^2 + 1 = 0$

$K_{1,2} = \pm i$

$Y(x, c_1, c_2) = c_1 \cos x + c_2 \sin x$

$p(x) = 4$

$q(x) = a \qquad q'(x) = q''(x) = 0$

$a = 4$

soluzione: $y = c_1 \cos x + c_2 \sin x + 4$

$y'' + y = x + 1$

$K^2 + 1 = 0$

$K = \pm i$

$Y(x, c_1, c_2) = c_1 \cos x + c_2 \sin x$

$p(x) = x + 1$

$q(x) = ax + b \qquad q'(x) = a \qquad q''(x) = 0$

$ax + b = x + 1 \rightarrow a = 1 \quad b = 1$

soluzione:

$y = c_1 \cos x + c_2 \sin x + x + 1$

$$y'' + 2y' + 2y = 6$$

$$K^2 + 2K + 2 = 0$$

$$\frac{\Delta}{4} = 1 - 2 = -1$$

$$K_{1,2} = -1 \pm i$$

$$Y(x, c_1, c_2) = e^{-x}\left(c_1 \cos x + c_2 \sin x\right)$$

$$p(x) = 6$$

$$q(x) = a \qquad q'(x) = q''(x) = 0$$

$$2a = 6 \rightarrow a = 3$$

soluzione:

$$y = e^{-x}\left(c_1 \cos x + c_2 \sin x\right) + 3$$

$$y'' + y' = x + 2$$

$$K^2 + K = 0$$

$$K(K+1) = 0$$

$$K_1 = 0, \quad K_2 = -1$$

$$Y(x, c_1, c_2) = c_1 + c_2 e^{-x}$$

$$p(x) = x + 2$$

$$q(x) = ax^2 + bx + c \qquad q'(x) = 2ax + b \qquad q''(x) = 2a$$

$$2a + 2ax + b = x + 2 \rightarrow \begin{cases} 2a = 1 \\ 2a + b = 2 \end{cases} \qquad \begin{cases} a = \frac{1}{2} \\ b = 1 \end{cases}$$

soluzione:

$$y = c_1 + c_2 e^{-x} + \frac{1}{2}x^2 + x$$

$$y'' + 4y = x^2$$

$$K^2 + 4 = 0$$

$$K_{1,2} = \pm 2i$$

$$Y(x, c_1, c_2) = c_1 \cos 2x + c_2 \sin 2x$$

$$p(x) = x^2$$

$$q(x) = ax^2 + bx + c \qquad q'(x) = 2ax + b \qquad q''(x) = 2a$$

$$2a + 4ax^2 + 4bx + 4c = x^2$$

$$\begin{cases} 4a = 1 \\ 4b = 0 \\ 2a + 4c = 0 \end{cases} \qquad \begin{cases} a = \frac{1}{4} \\ b = 0 \\ c = -\frac{1}{8} \end{cases}$$

soluzione: $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4}x^2 - \frac{1}{8}$

$$y'' + 6y' + 9y = 3(3x - 1)$$

$$K^2 + 6K + 9 = 0$$

$$(K+3)^2 = 0$$

$$K_{1,2} = -3$$

$$Y(x, c_1, c_2) = (c_1 + c_2 x) e^{-3x}$$

$$p(x) = 9x - 3$$

$$q(x) = ax + b \qquad q'(x) = a \qquad q''(x) = 0$$

$$6a + 9ax + 9b = 9x - 3$$

$$\begin{cases} 9a = 9 \\ 6a + 9b = -3 \end{cases} \qquad \begin{cases} a = 1 \\ b = -1 \end{cases}$$

soluzione:

$$y = (c_1 + c_2 x) e^{-3x} + x - 1$$

$y'' - y = 2x^2 + 1$

$K^2 - 1 = 0$

$K = \pm 1$

$Y(x, c_1, c_2) = c_1 e^{x} + c_2 e^{-x}$

$p(x) = 2x^2 + 1$

$q(x) = ax^2 + bx + c \qquad q'(x) = 2ax + b \qquad q''(x) = 2a$

$2a - ax^2 - bx - c = 2x^2 + 1$

$$\begin{cases} -a = 2 \\ -b = 0 \\ 2a - c = 1 \end{cases} \qquad \begin{cases} a = -2 \\ b = 0 \\ c = -5 \end{cases}$$

soluzione: $y = c_1 e^{x} + c_2 e^{-x} - 2x^2 - 5$

$y'' - 2y' = 5$

$y(0) = 0$

$y'(0) = 0$

$K^2 - 2K = 0$

$K = 0 \ \vee \ K = 2$

$Y(x, c_1, c_2) = c_1 + c_2 e^{2x}$

$p(x) = 5 \qquad q(x) = ax + b \qquad q'(x) = a \qquad q''(x) = 0$

$-2a = 5 \rightarrow a = -\frac{5}{2}$

$y = c_1 + c_2 e^{2x} - \frac{5}{2}x \qquad y' = 2c_2 e^{2x} - \frac{5}{2}$

$y(0) = c_1 + c_2 = 0 \qquad y'(0) = 2c_2 - \frac{5}{2} = 0$

$$\begin{cases} c_1 + c_2 = 0 \\ 2c_2 - \frac{5}{2} = 0 \end{cases} \qquad \begin{cases} c_1 = -\frac{5}{4} \\ c_2 = \frac{5}{4} \end{cases}$$

Soluzione:

$$y = \frac{5}{4} e^{2x} - \frac{5}{2} x - \frac{5}{4}$$

$$y'' + 4y = 4x^2 + 4x + 6$$

$$K^2 + 4 = 0$$

$$K_{1,2} = \pm 2i$$

$$Y(x, c_1, c_2) = c_1 \cos 2x + c_2 \sin 2x$$

$$p(x) = 4x^2 + 4x + 6$$

$$q(x) = ax^2 + bx + c \qquad q'(x) = 2ax + b \qquad q''(x) = 2a$$

$$2a + 4ax^2 + 4bx + 4c = 4x^2 + 4x + 6$$

$$\begin{cases} 4a = 4 \\ 4b = 4 \\ 2a + 4c = 6 \end{cases} \qquad \begin{cases} a = 1 \\ b = 1 \\ c = 1 \end{cases}$$

Soluzione: $y = c_1 \cos 2x + c_2 \sin 2x + x^2 + x + 1$

$$y'' - y = e^{-x}$$

$$K^2 - 1 = 0$$

$$K = \pm 1$$

$$Y(x, c_1, c_2) = c_1 e^{-x} + c_2 e^{x}$$

$$p(x) = e^{-x}$$

$$q(x) = Bxe^{-x}$$

$$q'(x) = Be^{-x}(1 - x)$$

$$q''(x) = Be^{-x}(x - 2)$$

$$Be^{-x}(x - 2 - x) = e^{-x} \rightarrow B = -\frac{1}{2}$$

soluzione: $y = \left(c_1 - \frac{1}{2}x\right)e^{-x} + c_2 e^{x}$

$$y'' - 3y' + 2y = e^x$$

$$K^2 - 3K + 2 = 0$$

$$K = \frac{3 \pm 1}{2} = \begin{cases} 1 \\ 2 \end{cases}$$

$$Y(x, c_1, c_2) = c_1 e^x + c_2 e^{2x}$$

$$p(x) = e^x$$

$$q(x) = Bxe^x$$

$$q'(x) = Be^x(1+x)$$

$$q''(x) = Be^x(x+2)$$

$$Be^x(x+2-3-3x+2x) = e^x$$

$$-B = 1 \quad \rightarrow \quad B = -1$$

soluzione : $y = c_1 e^x + c_2 e^{2x} - xe^x$

$$y'' + y = e^x$$

$$K^2 + 1 = 0$$

$$K = \pm i$$

$$Y(x, c_1, c_2) = c_1 \cos x + c_2 \sin x$$

$$p(x) = e^x$$

$$q(x) = Be^x$$

$$q'(x) = q''(x) = Be^x$$

$$Be^x + Be^x = e^x$$

$$2B = 1 \quad \rightarrow \quad B = \frac{1}{2}$$

soluzione : $y = c_1 \cos x + c_2 \sin x + \frac{1}{2} e^x$

$$y'' - 2y' + y = 2e^x$$

$$K^2 - 2K + 1 = 0$$

$$K_{1,2} = 1$$

$$Y(x, c_1, c_2) = (c_1 + c_2 x)\, e^x$$

$$p(x) = 2e^x \qquad q(x) = Bx^2 e^x$$

$$q'(x) = 2Bx e^x + Bx^2 e^x = Be^x(2x + x^2)$$

$$q''(x) = Be^x(2x + x^2 + 2 + 2x) = Be^x(x^2 + 4x + 2)$$

$$Be^x(\cancel{x^2} + \cancel{4x} + 2 \cancel{-4x} \cancel{-2x^2} \cancel{+x^2}) = 2e^x$$

$$2B = 2 \rightarrow B = 1$$

soluzione : $y = (c_1 + c_2 x + x^2)\, e^x$

$$y'' - y = \sin x$$

$$K^2 - 1 = 0$$

$$K = \pm 1$$

$$Y(x, c_1, c_2) = c_1 e^x + c_2 e^{-x}$$

$$p(x) = \sin x \qquad q(x) = A \sin x + B \cos x$$

$$q'(x) = A \cos x - B \sin x$$

$$q''(x) = -A \sin x - B \cos x$$

$$-A \sin x - B \cos x - A \sin x - B \cos x = \sin x$$

$$\begin{cases} -2A = 1 \\ -2B = 0 \end{cases} \qquad \begin{cases} A = -\frac{1}{2} \\ B = 0 \end{cases}$$

soluzione : $y = c_1 e^x + c_2 e^{-x} - \frac{1}{2} \sin x$

$$y'' - 3y' = \sin x + \cos x$$

$$p(x) = \sin x + \cos x$$

$$q(x) = A \sin x + B \cos x$$

$$q'(x) = A \cos x - B \sin x$$

$$q''(x) = -A \sin x - B \cos x$$

$$K^2 - 3K = 0$$

$$K = 0 \quad \vee \quad K = 3$$

$$Y(x, c_1, c_2) = c_1 + c_2 e^{3x}$$

$$-A \sin x - B \cos x - 3A \cos x + 3B \sin x = \sin x + \cos x$$

$$\begin{cases} -A + 3B = 1 \\ -B - 3A = 1 \end{cases} \qquad \begin{cases} A = 3B - 1 \\ -B - 9B + 3 = 1 \end{cases} \qquad \begin{cases} A = -\frac{2}{5} \\ B = \frac{1}{5} \end{cases}$$

soluzione : $y = c_1 + c_2 e^{3x} - \frac{2}{5} \sin x + \frac{1}{5} \cos x$

$$4y'' + 9y = 5\sin x$$

$$y(0) = 0$$

$$y'(0) = \frac{5}{2}$$

$$4K^2 + 9 = 0$$

$$K = \pm \frac{3}{2} i$$

$$Y(x, c_1, c_2) = c_1 \cos\frac{3}{2}x + c_2 \sin\frac{3}{2}x$$

$$p(x) = 5\sin x$$

$$q(x) = A\sin x + B\cos x$$

$$q'(x) = A\cos x - B\sin x$$

$$q''(x) = -A\sin x - B\cos x$$

$$-4A\sin x - 4B\cos x + 9A\sin x + 9B\cos x = 5\sin x$$

$$\begin{cases} 5A = 5 \\ 5B = 0 \end{cases} \qquad \begin{cases} A = 1 \\ B = 0 \end{cases}$$

$$y = c_1 \cos\frac{3}{2}x + c_2 \sin\frac{3}{2}x + \sin x$$

$$y' = -\frac{3}{2}c_1 \sin\frac{3}{2}x + \frac{3}{2}c_2 \cos\frac{3}{2}x + \cos x$$

$$y(0) = c_1 = 0$$

$$y'(0) = \frac{3}{2}c_2 + 1 = \frac{5}{2} \quad \rightarrow \quad c_2 = 1$$

soluzione: $y = \sin\frac{3}{2}x + \sin x$